Colouring Non-Even Digraphs

Raphael Steiner

joint work with Marcelo Garlet Millani and Sebastian Wiederrecht

Graph Colouring

G graph, $c: V(G) \to \{1, \dots, k\}$ proper colouring

$\Leftrightarrow$

$c(x) \neq c(y)$ for all $xy \in E(G)$

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Conjecture

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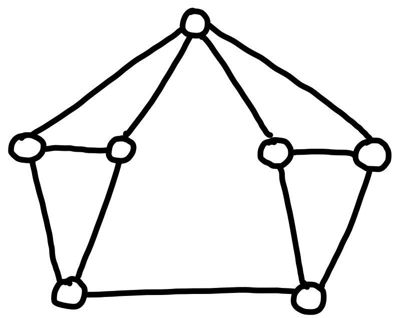

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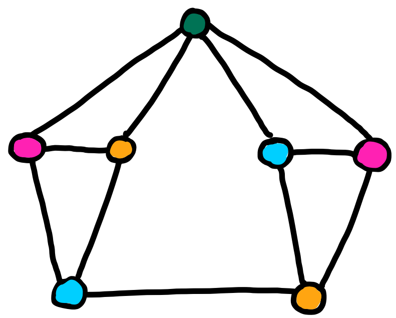

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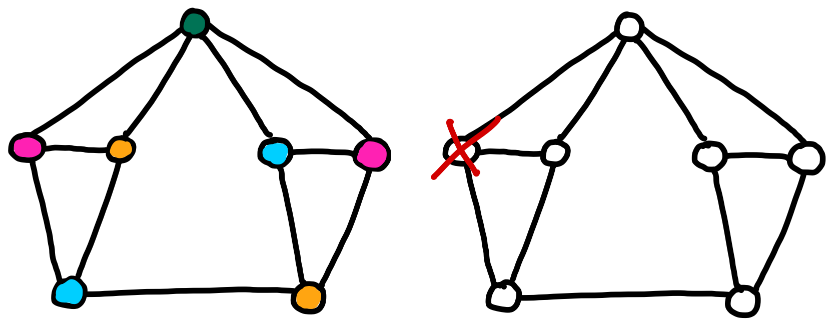

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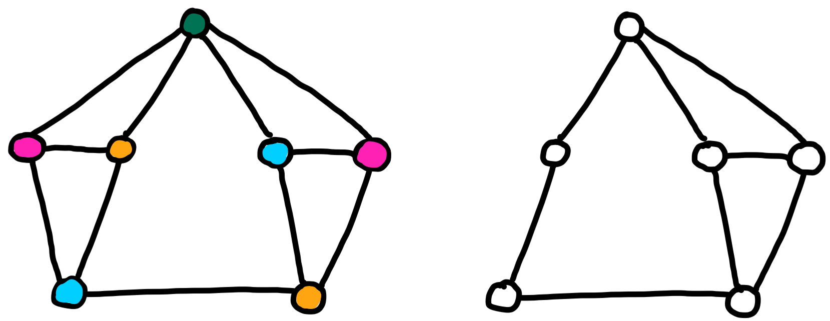

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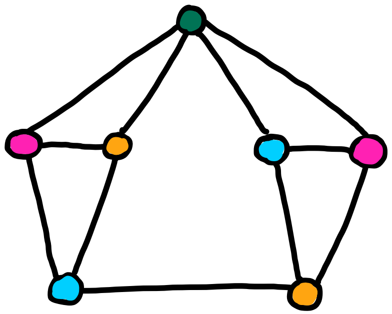

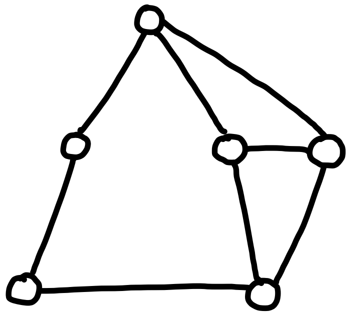

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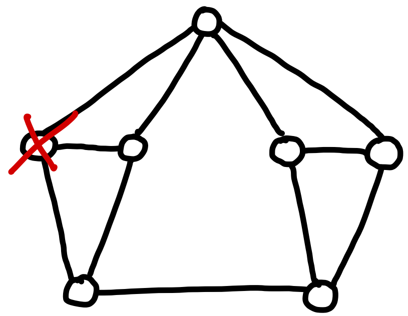

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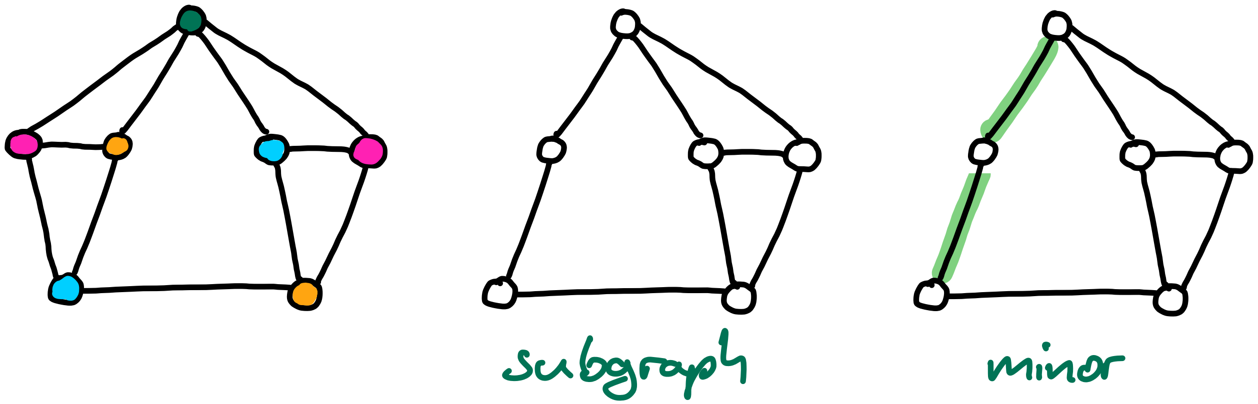

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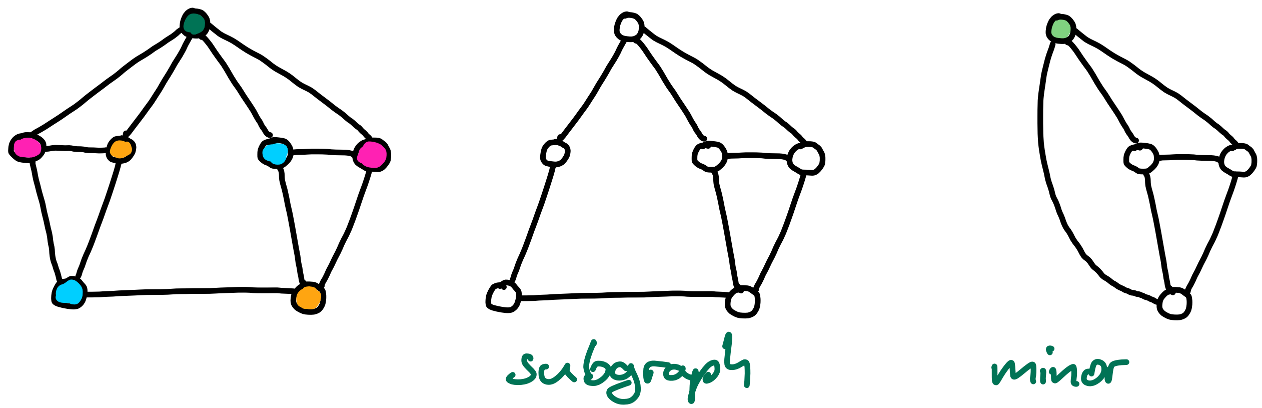

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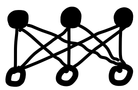
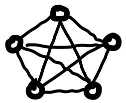
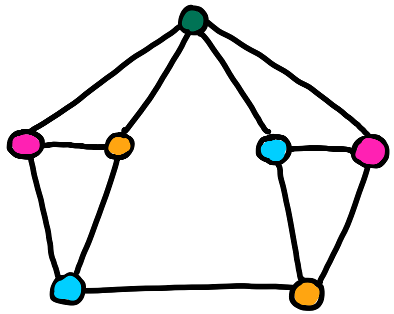
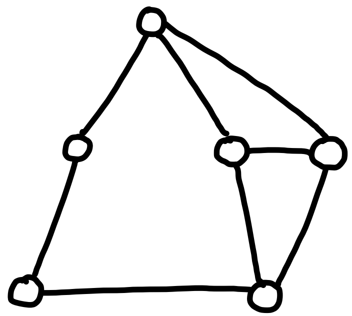
G planar $\iff$ no  

K_5





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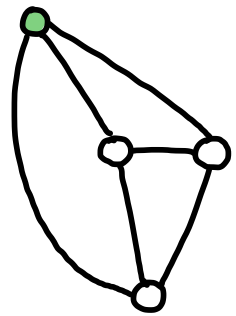

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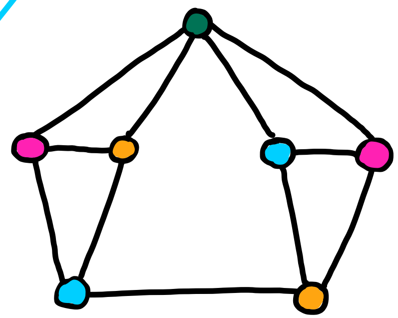

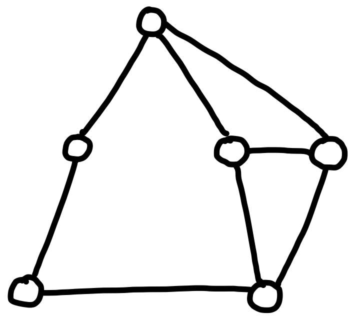

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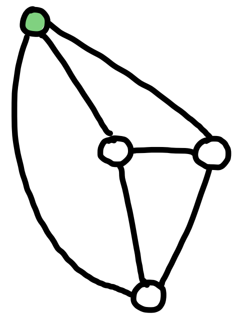

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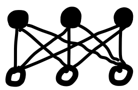
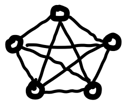
G planar $\Longleftrightarrow$ no  

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Hadwiger's Conjecture

$\chi(G) \geq n \Rightarrow K_n$-minor

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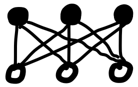
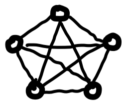
G planar $\iff$ no  

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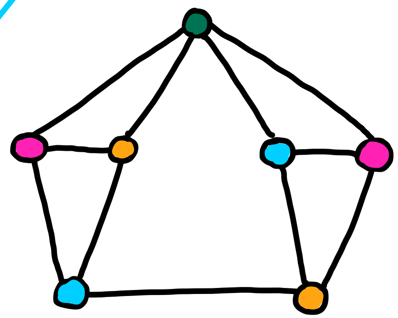

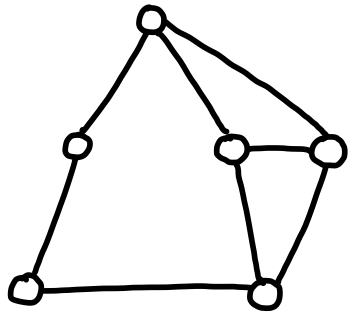

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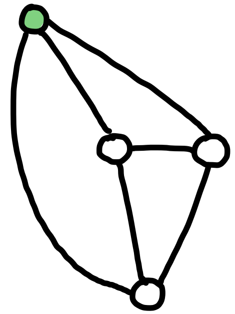

minor

__Hadwiger's Conjecture__

$\chi(G) \geq n \implies K_n$-minor

"One of the deepest problems in Combinatorics."
– P. Erdős

The Dichromatic Number

Introduced by Neumann-Lara in 1982

Goal: Generalise proper colourings for undirected graphs to digraphs in a meaningful way

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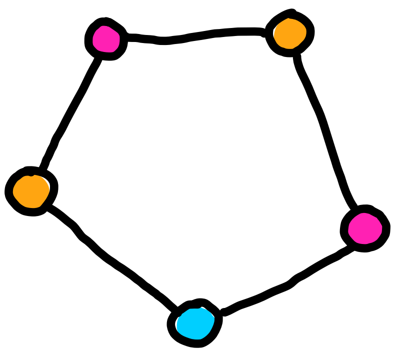

undirected: adjacent vertices must have different colours

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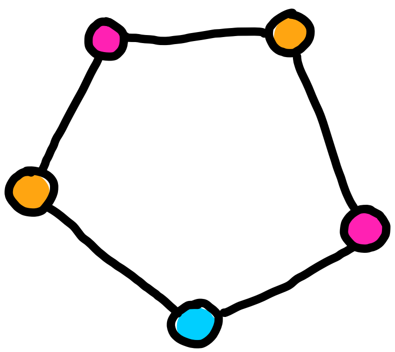

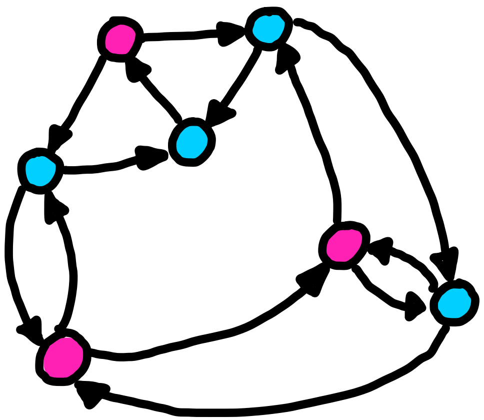

undirected: adjacent vertices must have different colours

$\chi(G)$

directed: no monochromatic directed cycles

$\vec{\chi}(D)$

A Conjecture

Two Colour Conjecture (Erdős, Neumann-Lara, Škrekovski)

Every orientation D of a planar graph is 2-colourable.

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We will talk about bicolourings today.

A Negative Result

$\mathcal{D}$ digraph

$$\mathcal{C}_{\mathcal{D}} := \left(V(\mathcal{D}), \left\{ V(C) \mid C \underset{\text{cycle}}{\subseteq} \mathcal{D} \text{ dir.} \right\}\right)$$

cycle hypergraph of $\mathcal{D}$

A Negative Result

similarities between digraph colourings and hypergraph colourings

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$\Rightarrow$ hard even on digraphs of bounded directed treewidth

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Graphs

no odd cycles (bipartite)

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$\iff$

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nicely structured classes implying polynomial time algorithms

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Can we have a similar picture?

Ingredients:

- a notion of "substructure"
- odd cycles
- matching vs. vertex cover
- colours

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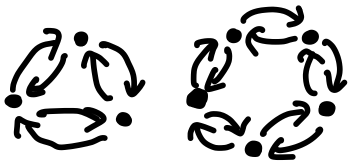 ...

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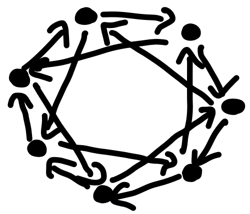

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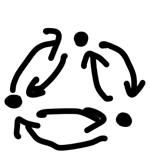
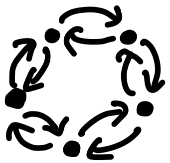
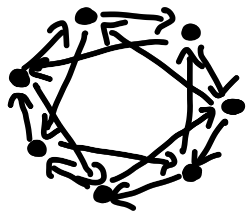
  … nor  as a butterfly minor

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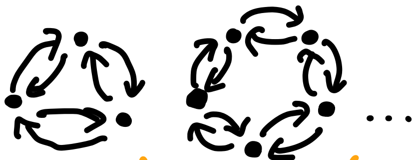

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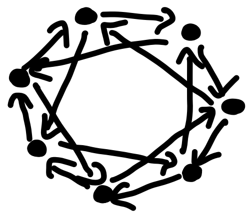
F_7



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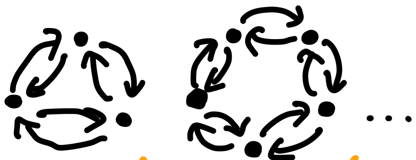

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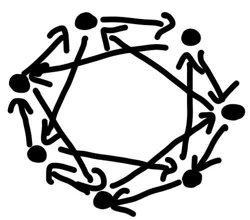

as a butterfly minor

dir. feedback vertex number

can be replaced by butterfly minor

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submitted 2001

the proof uses matching theory

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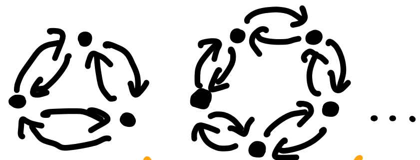

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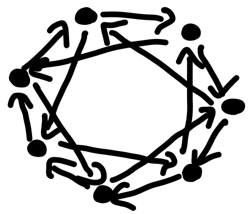

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What About Digraphs ? ⋰

Can we have a similar picture?

Ingredients:

- a notion of "substructure" butterfly minor
- odd cycles odd bicycles $(+ F_7)$
- matching vs. vertex cover cycle packing vs. feedback vertex set
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Theorem (Guenin & Thomas, 2011)

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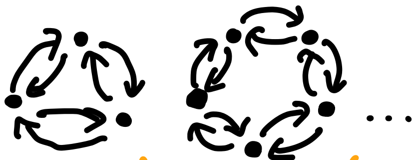

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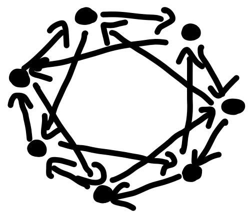

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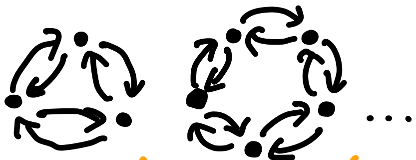

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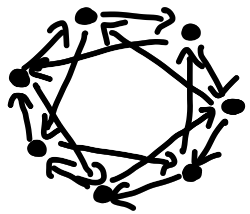

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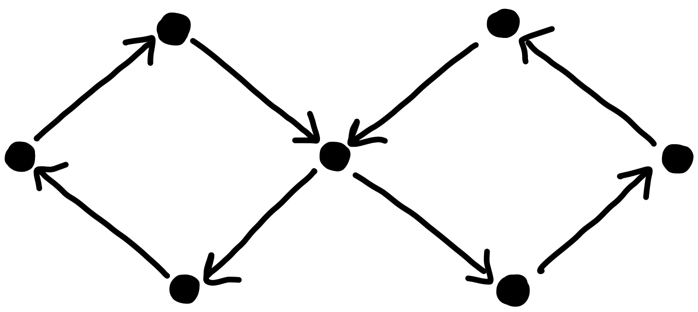
Digraphs

Bipartite with Perfect Matching



D

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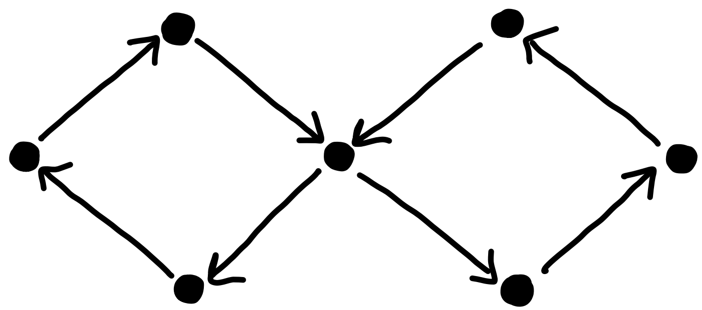

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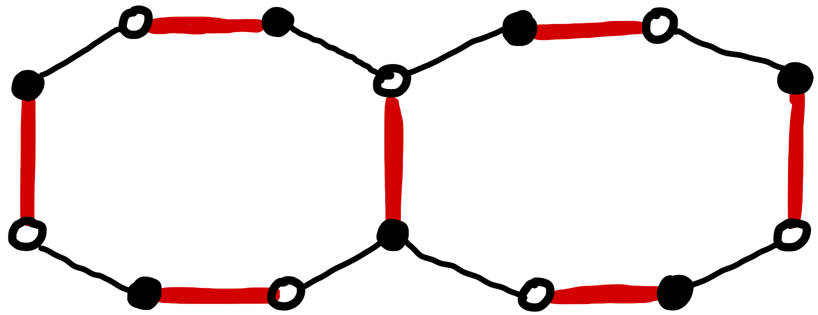

G, M

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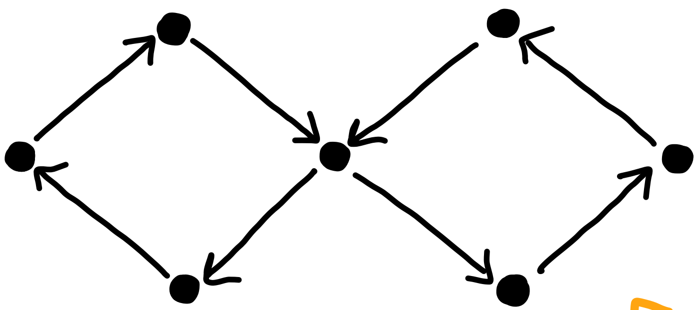

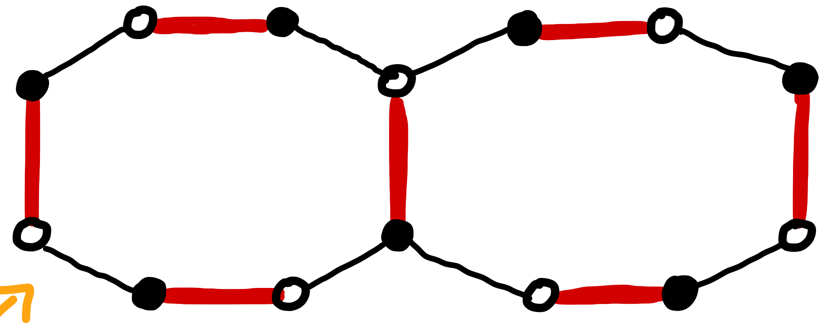

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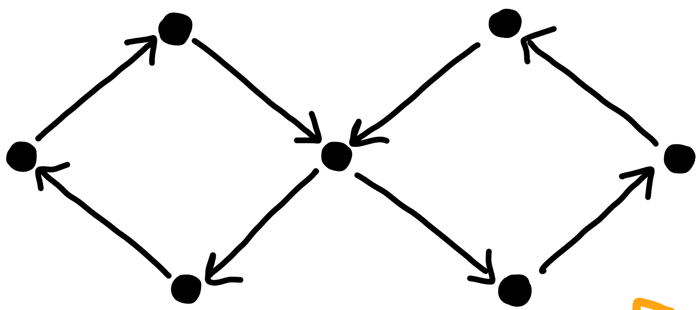

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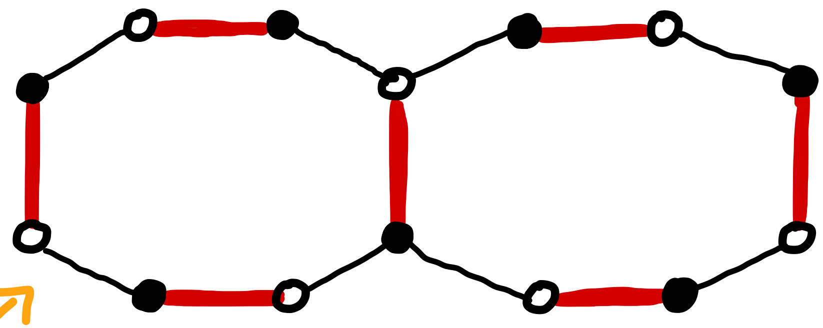

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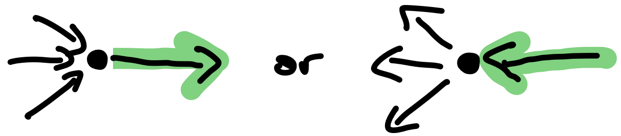
- contract 'special' edges:



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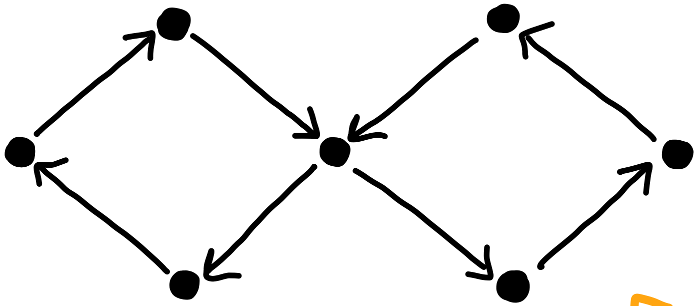

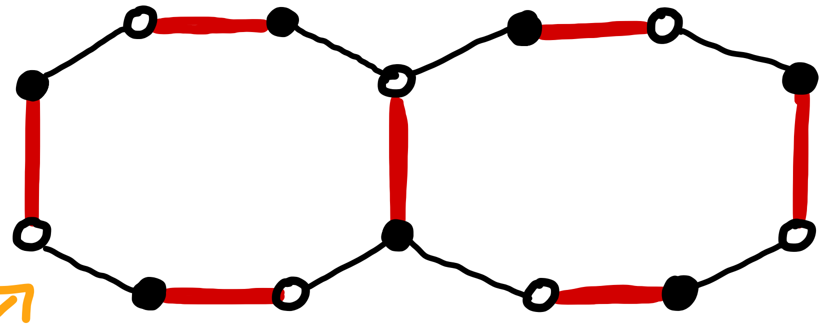

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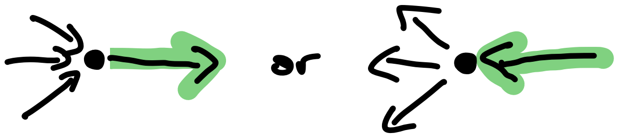

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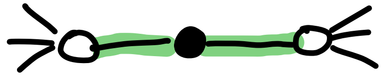

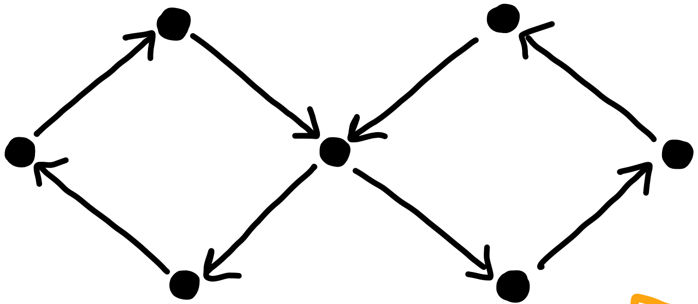

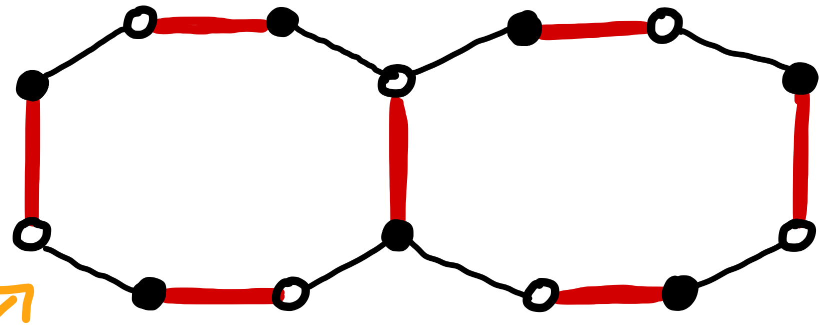

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leaving out some details

$\Leftrightarrow$

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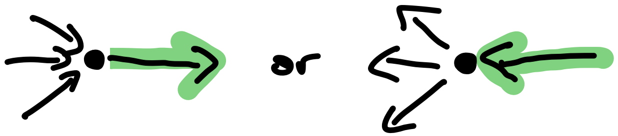

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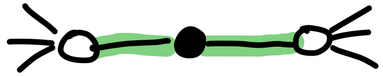

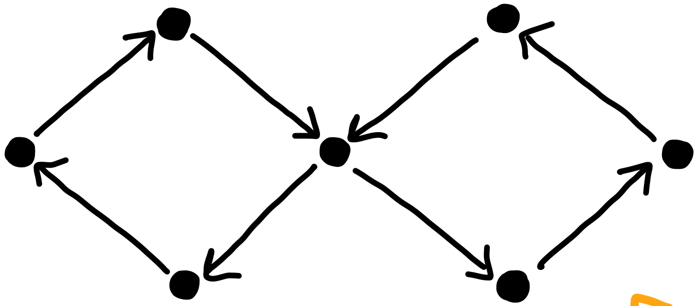

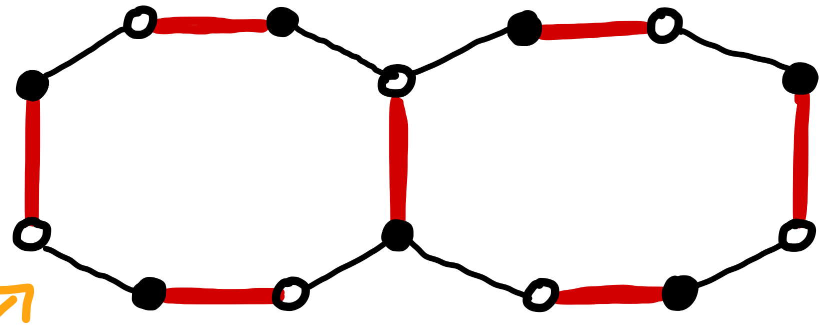

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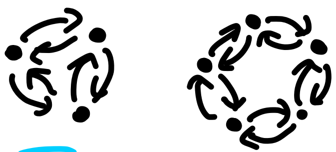 ...

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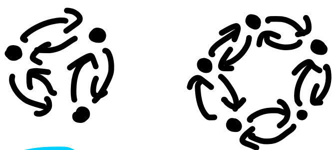
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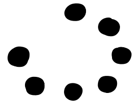
Pfaffian Graphs



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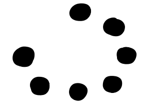

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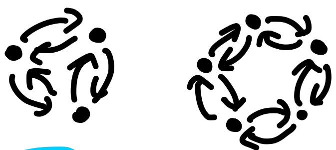

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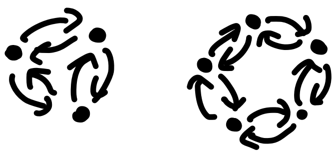

...

Little

$\Longleftrightarrow$

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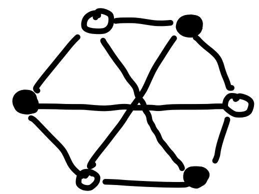

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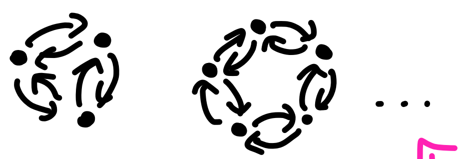

infinite antichain for butterfly minors

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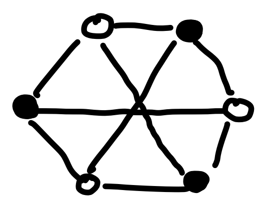

→ matching minors allow for more freedom than butterfly minors

Impressions of a Proof

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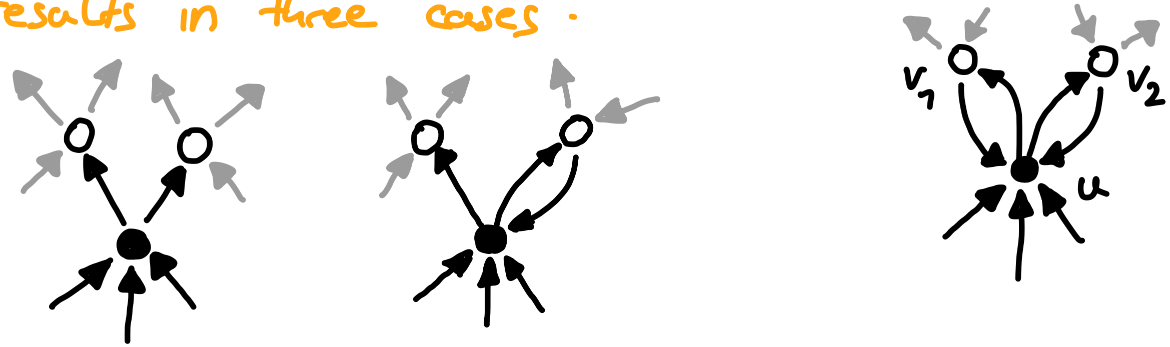

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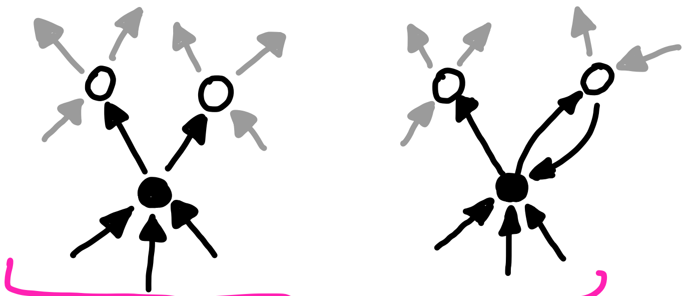

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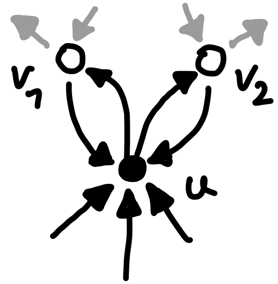

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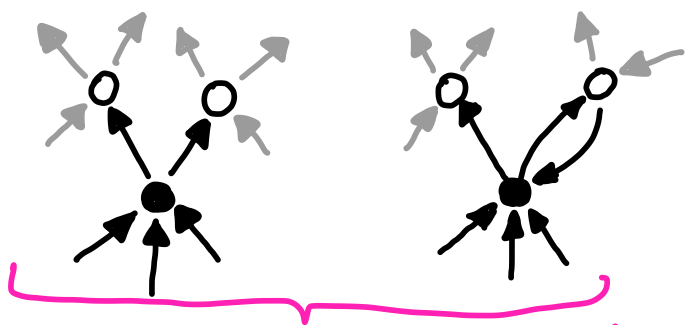

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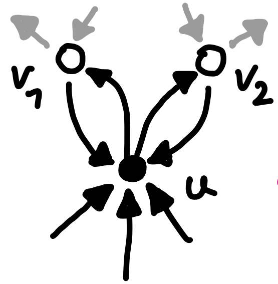

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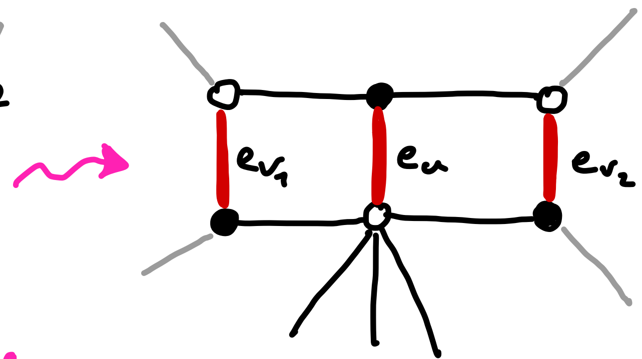

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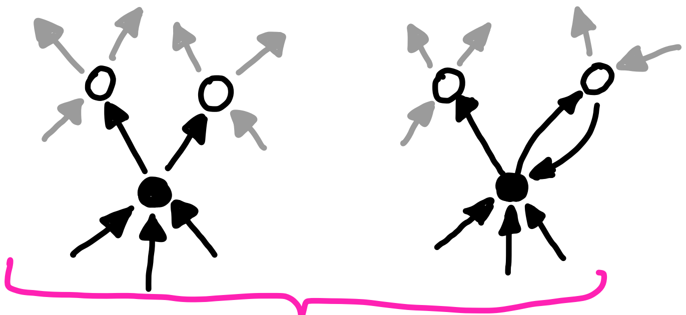

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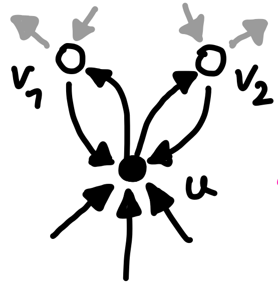

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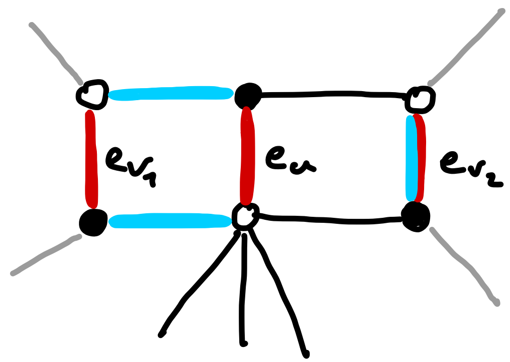

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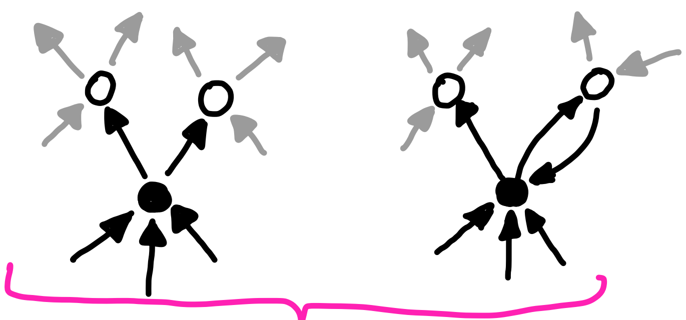

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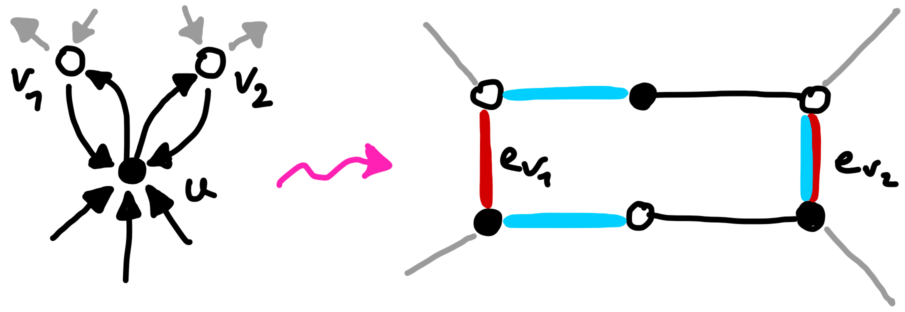

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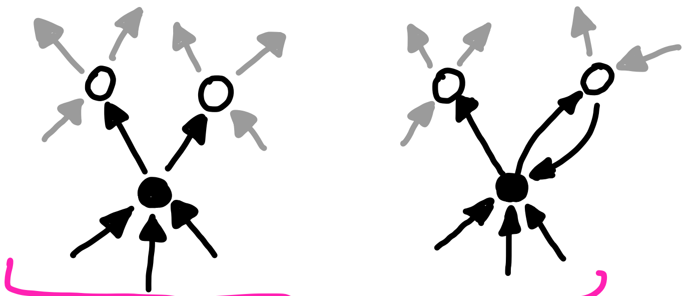

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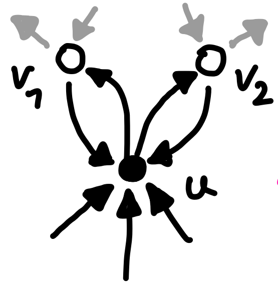

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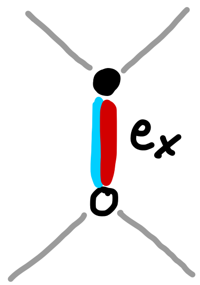

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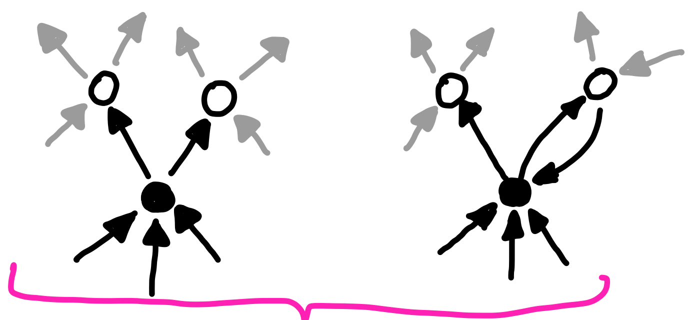

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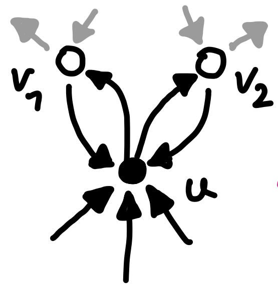

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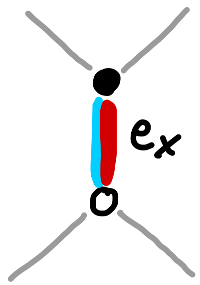

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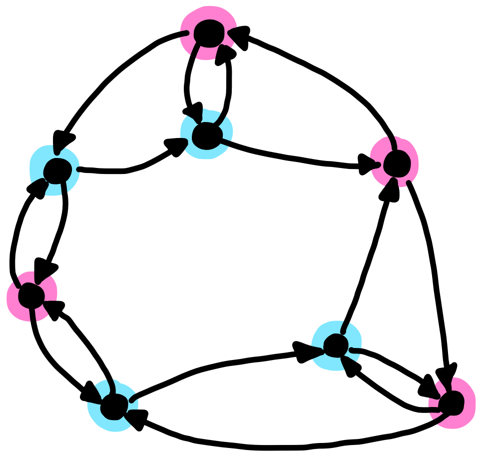

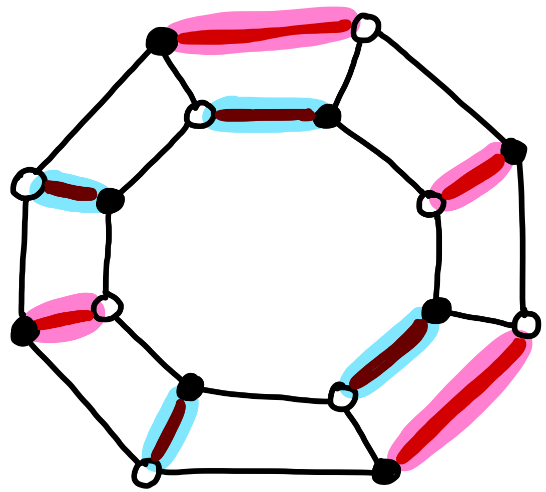

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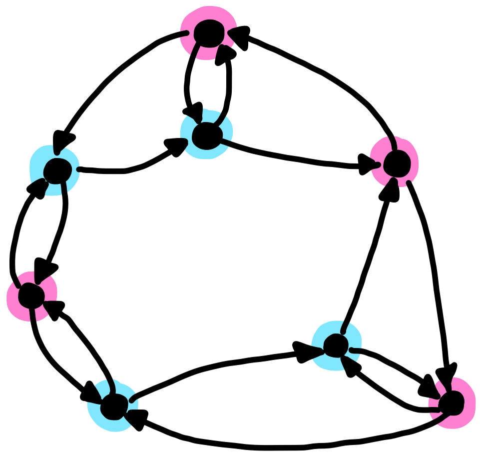

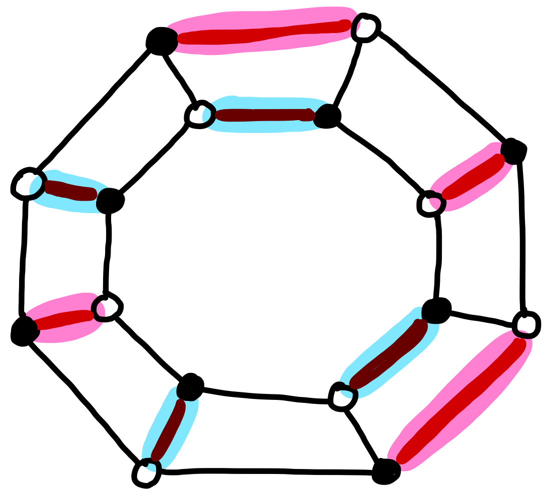

Another Conjecture

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Thank You