Current Challenges in Graph Minors

Sebastian Wiederrecht

Planar Graphs

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amazing algorithmic properties

- can be easily recognised
- MAXIMUM CUT [Hadlock 1975]
- #PERFECT MATCHINGS [FKT-algorithm 1960s]
- various routing problems

Excluding K_5

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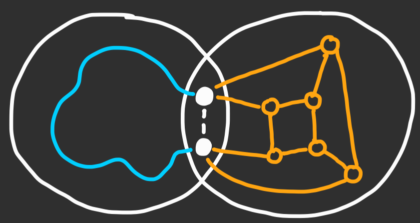
- 

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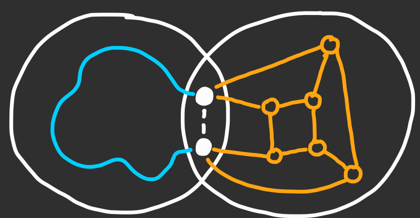
- 

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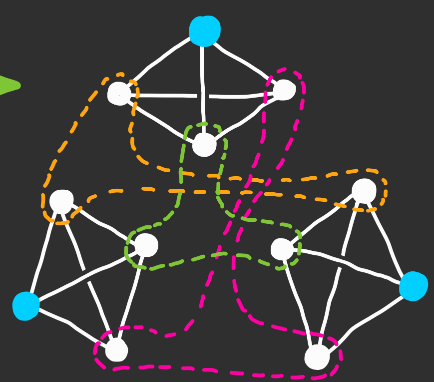
- 

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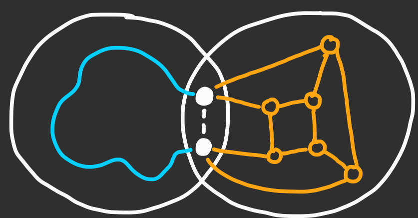
- 

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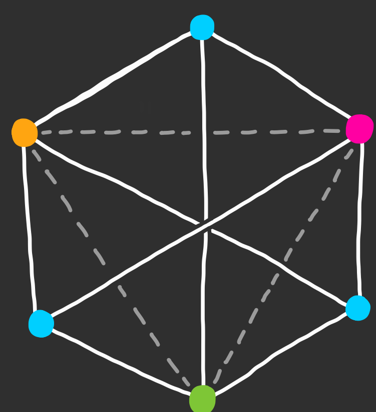
- 

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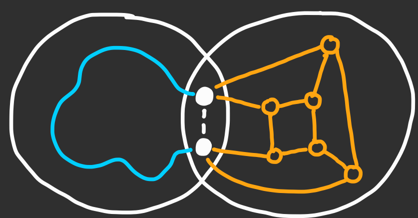
- 

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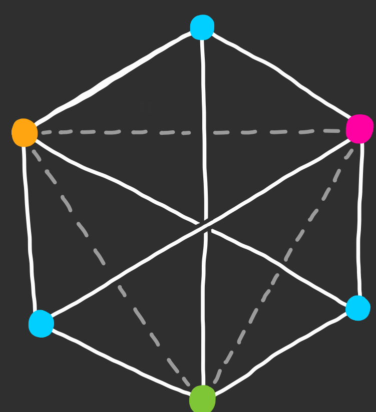
- 

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Theorem [Wagner 1937]

A graph is K_5-minor-free if and only if it can be constructed from planar graphs and  by means of ≤ 3-clique sums.

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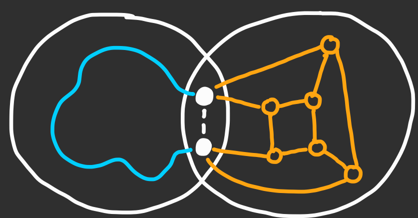
- 

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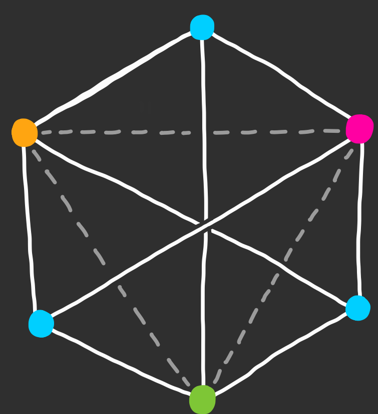
- 

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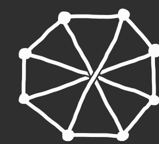
A graph is K_5-minor-free if and only if it can be constructed from planar graphs and  by means of ≤ 3-clique sums.

K_5-minor-free graphs

i) have a tree like structure

ii) are 4-colourable

iii) inherit many algorithmic properties of planar graphs

WQO?

Hadwiger's Conjecture

for which H does this continue?

Graph Minors: Robertson & Seymour-Style

two new parodigms

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   approximate instead of exact

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Theorem [R&S 1986] Grid Theorem

A minor-closed graph class $\mathcal{C}$ has bounded treewidth if and only if it does not contain all planar graphs.

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for all graphs H, any "area" of large treewidth in an H-minor-free graph is "essentially" planar

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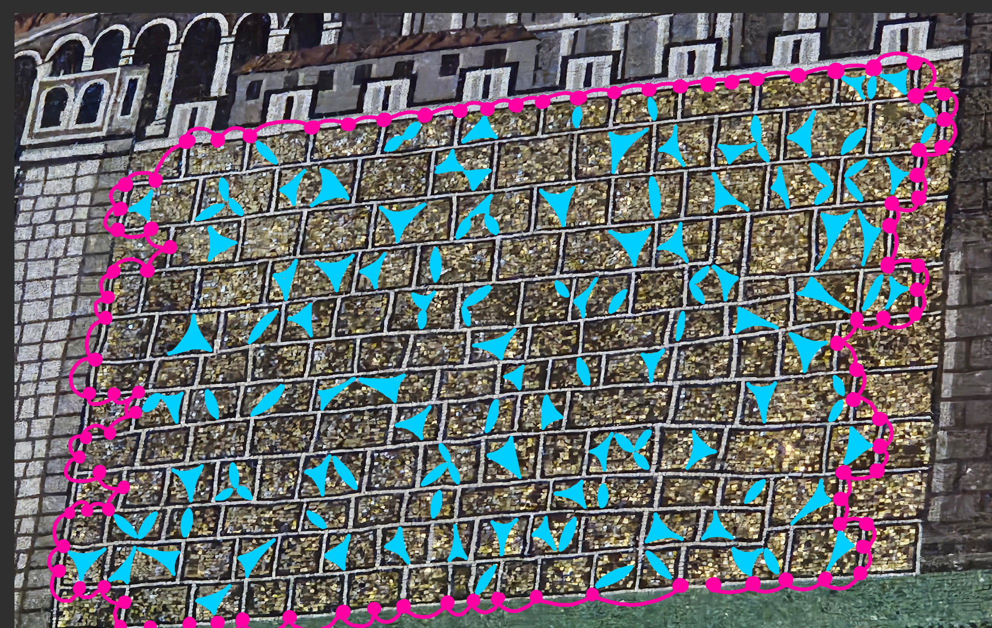

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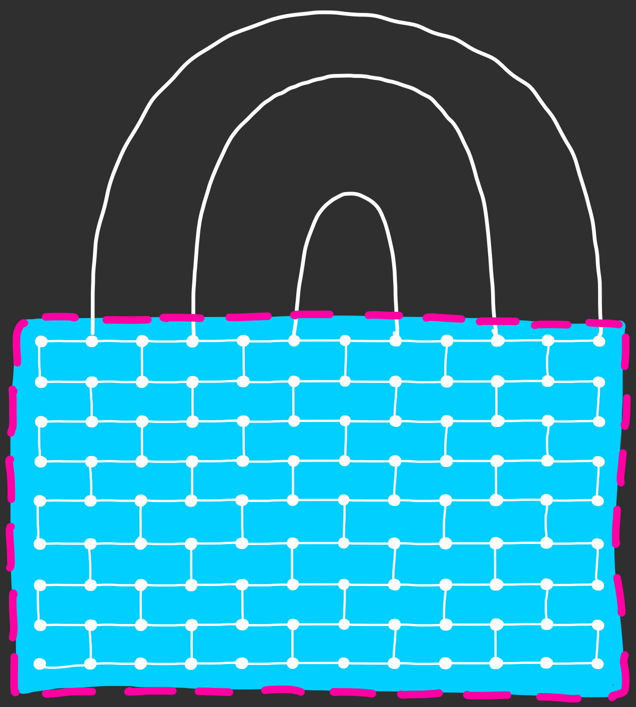

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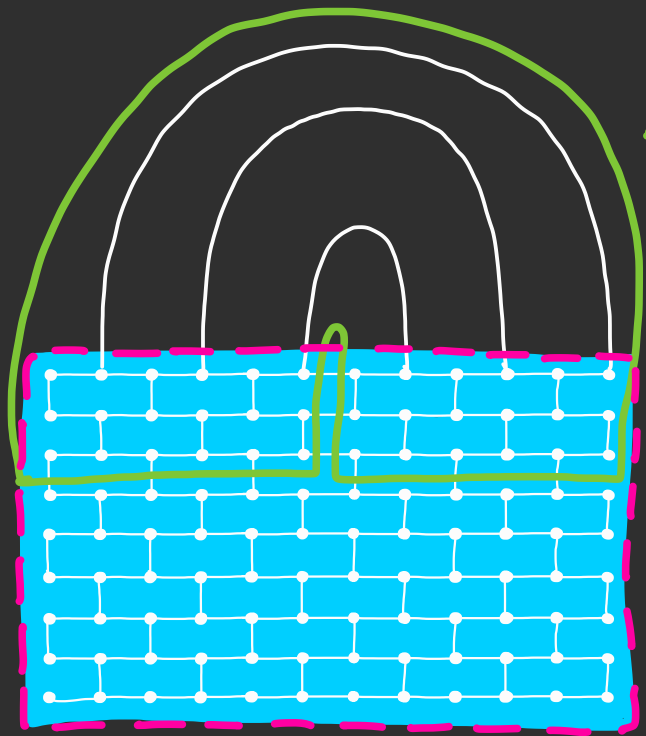

wall again ⇒ can be made flat

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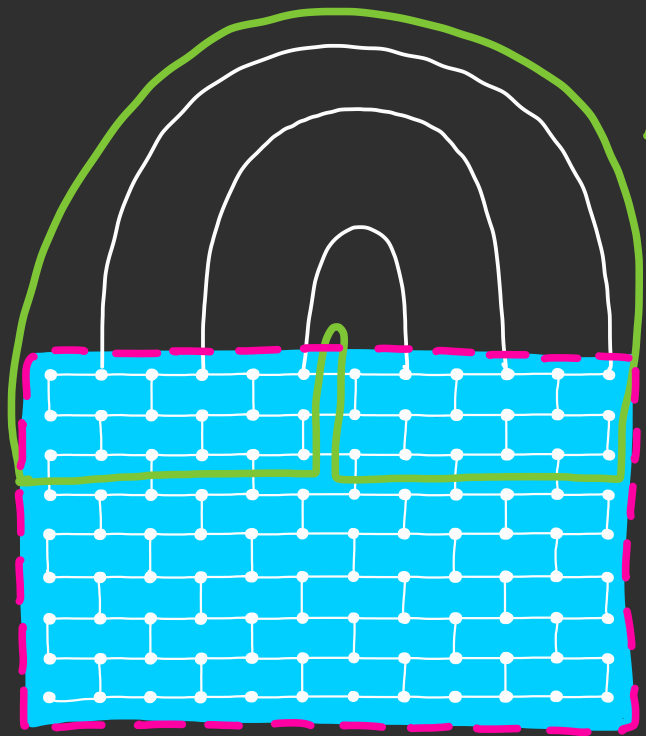

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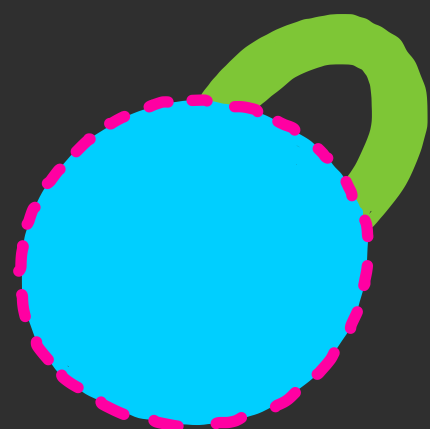

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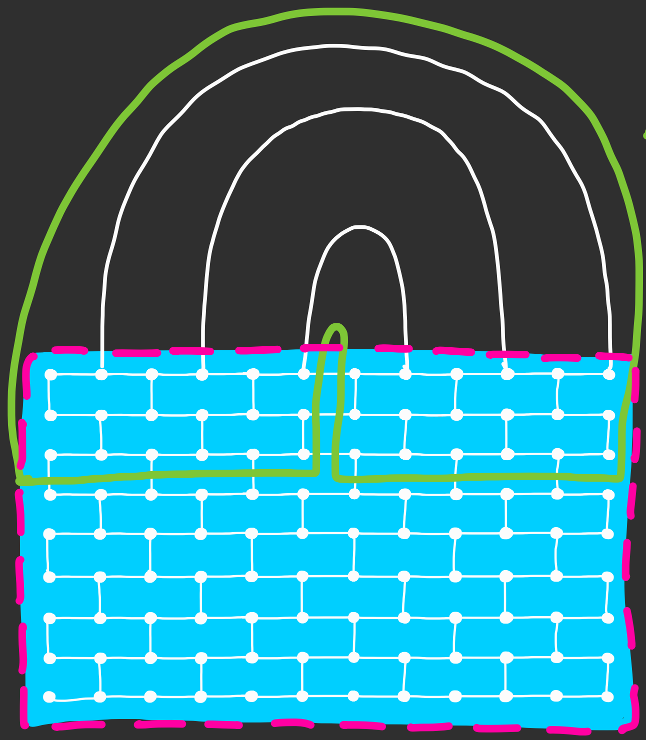

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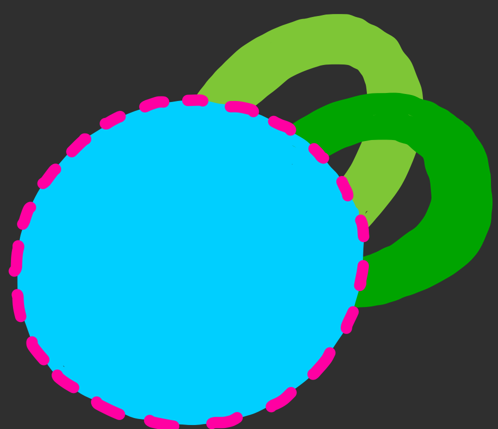

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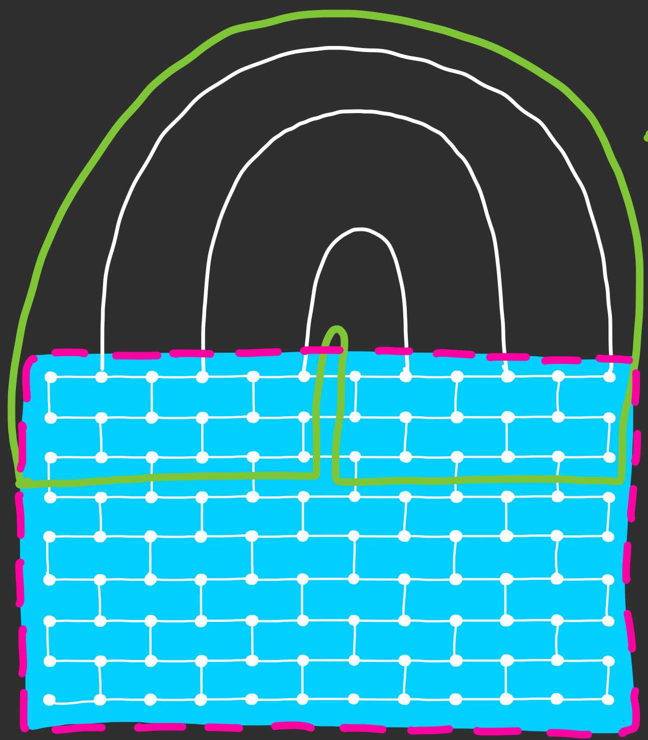

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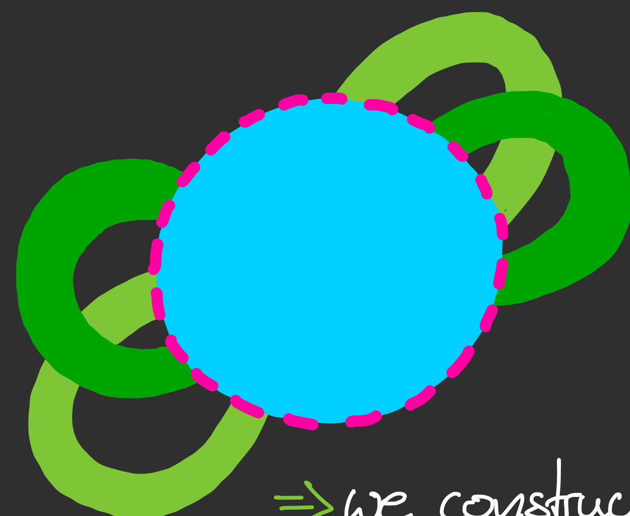

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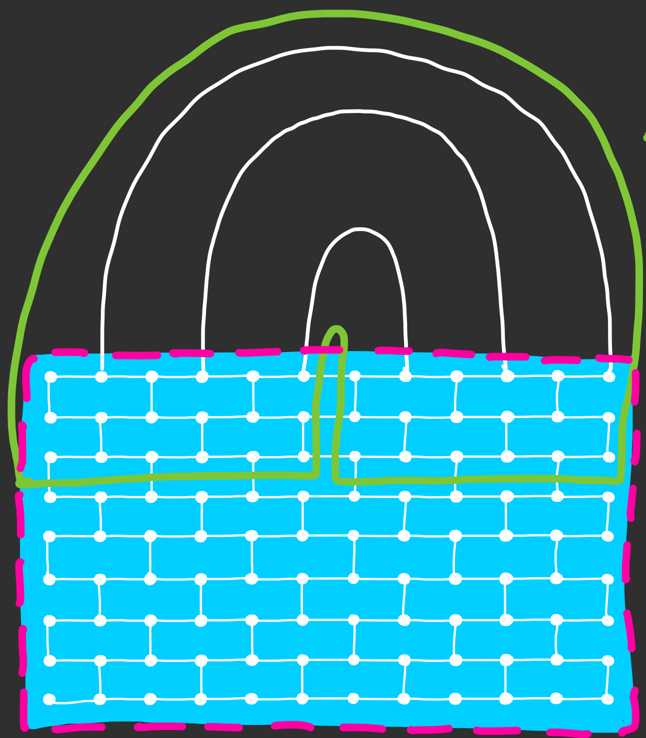

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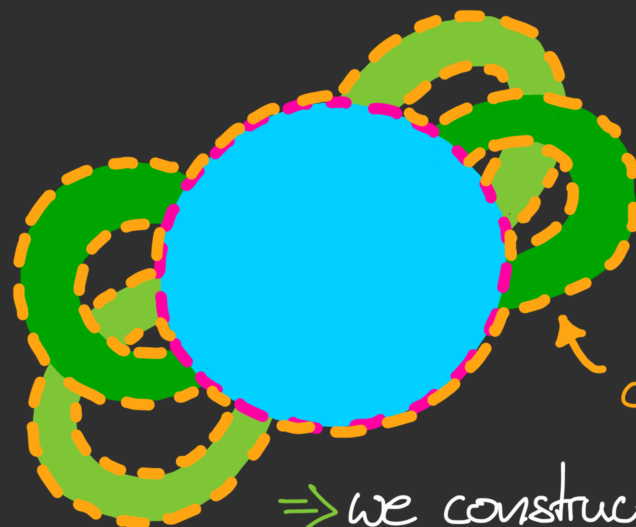

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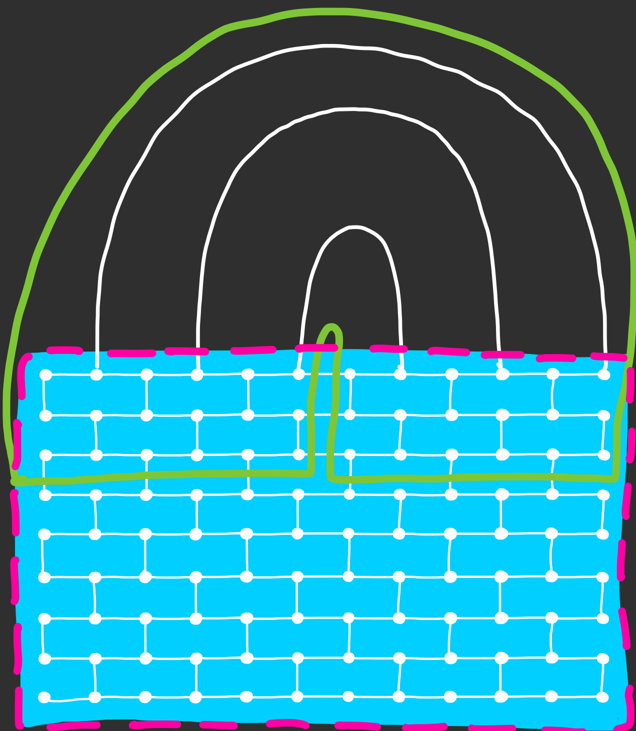

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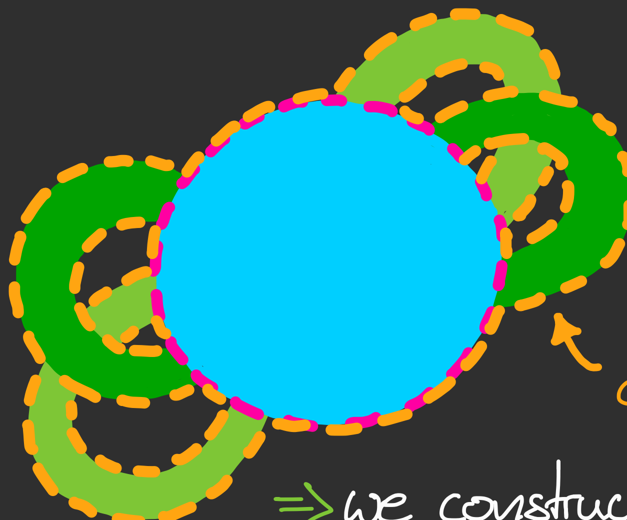

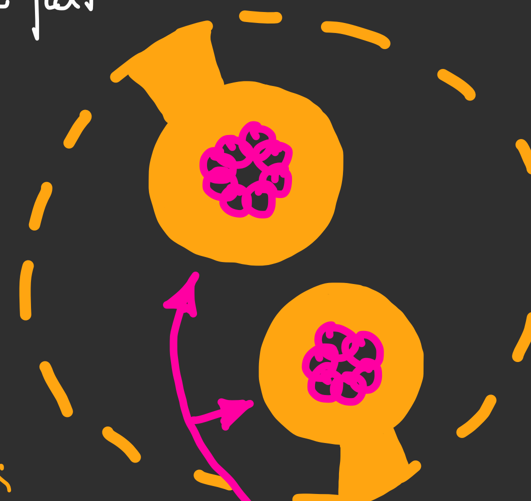

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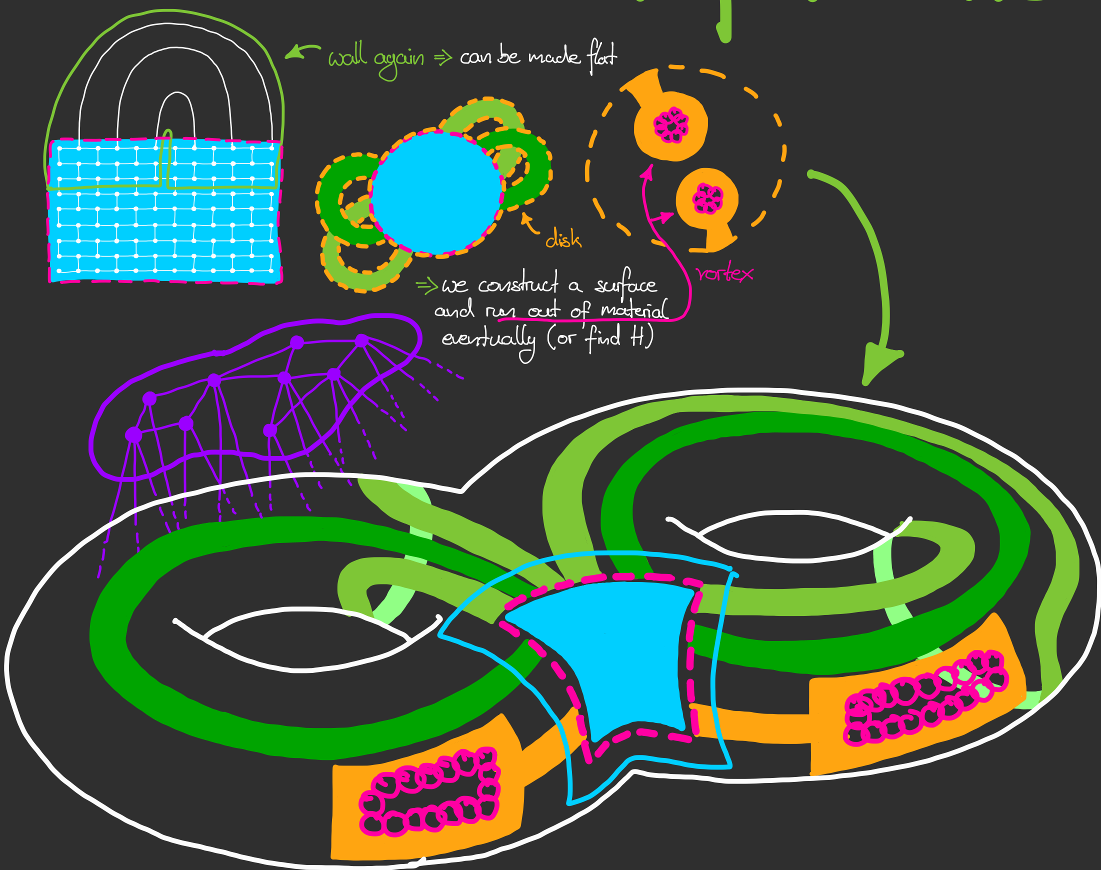

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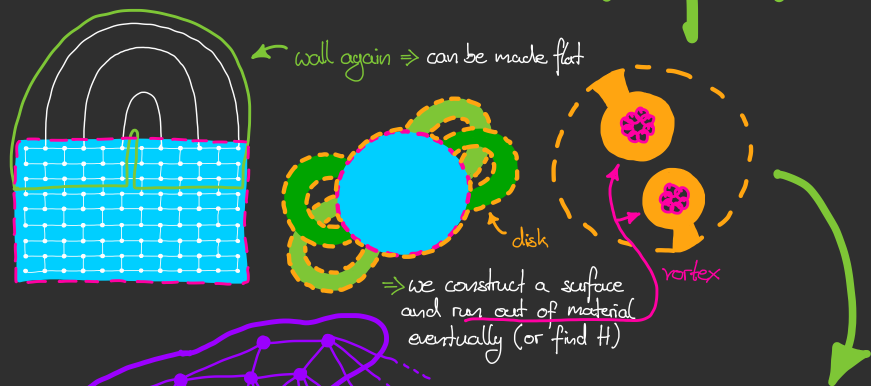

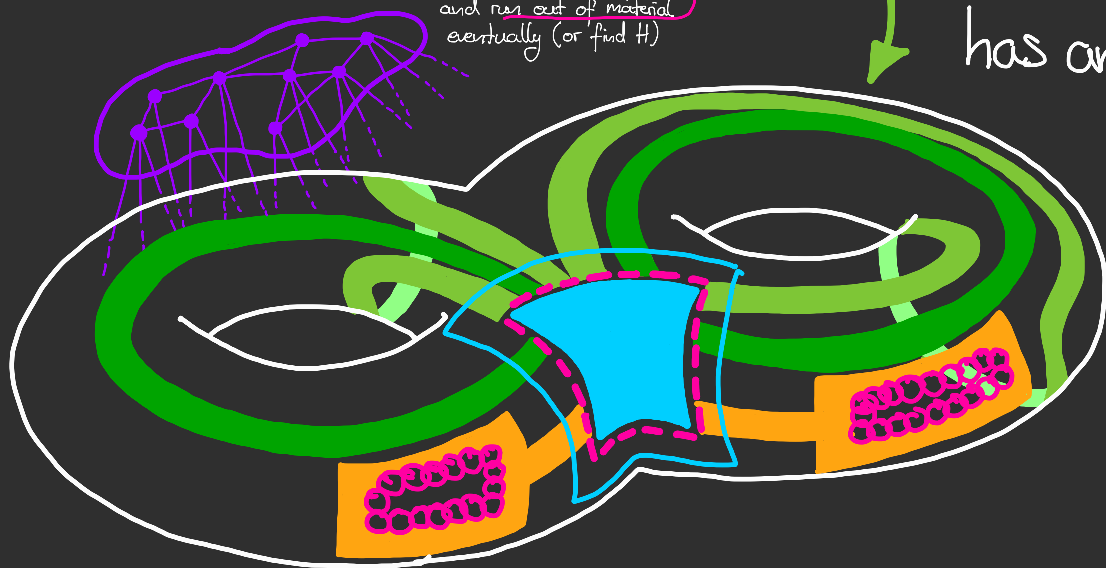

Local Structure Theorem [R&S 2003] $t := |V(H)|$

There are functions f, g s.t. every graph without an H-minor but with a $g(t)$-wall W has an almost embedding relative to W s.t.

- H does not embed in Σ
- there are at most $f(t)$ apices
- there are at most $O(t^2)$ vortices, each of depth at most $f(t)$

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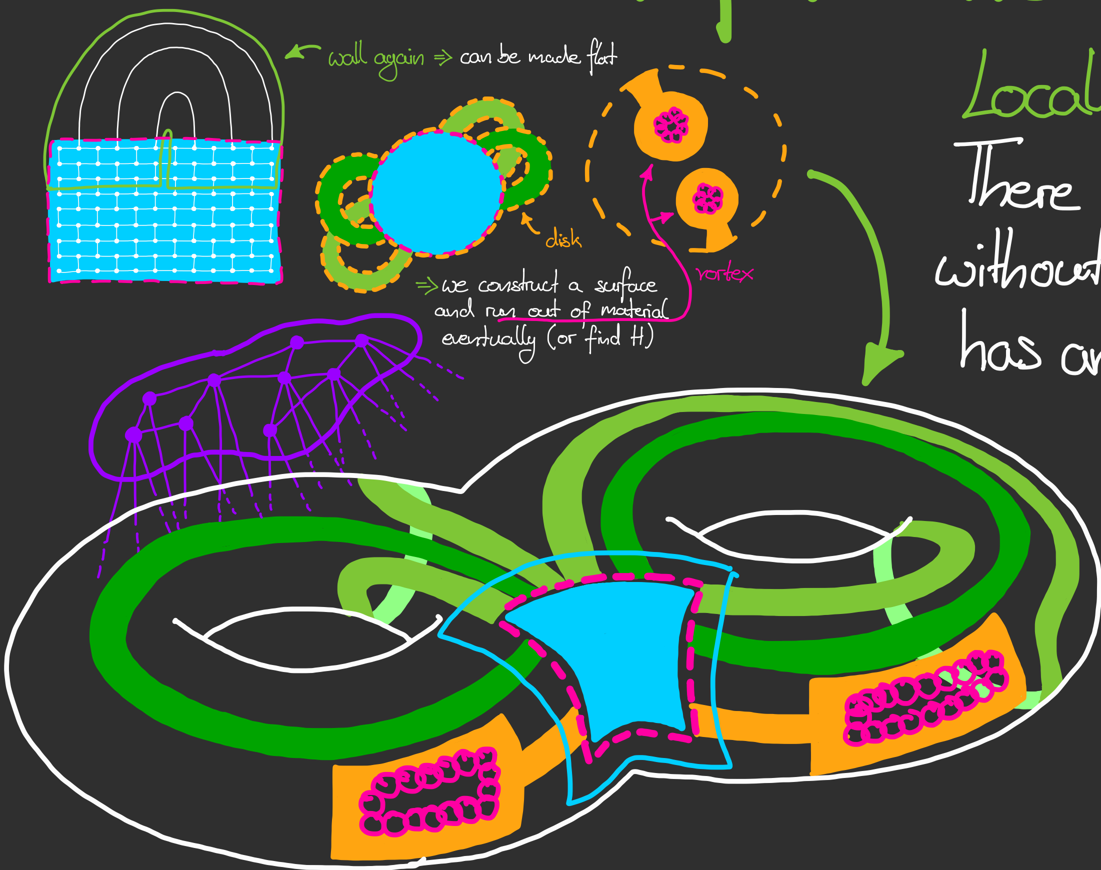

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$g(t), f(t) \in O(t^{115})$

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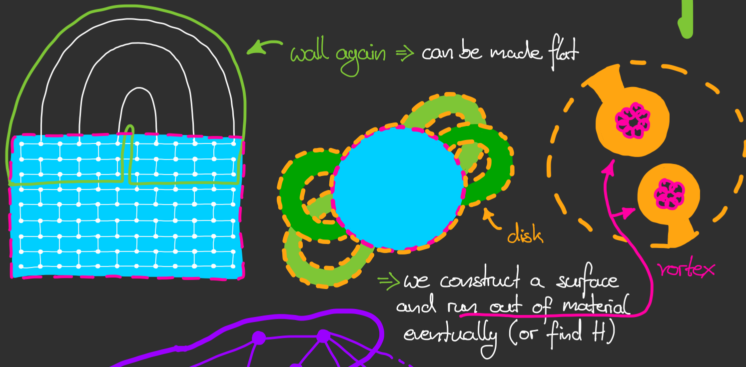

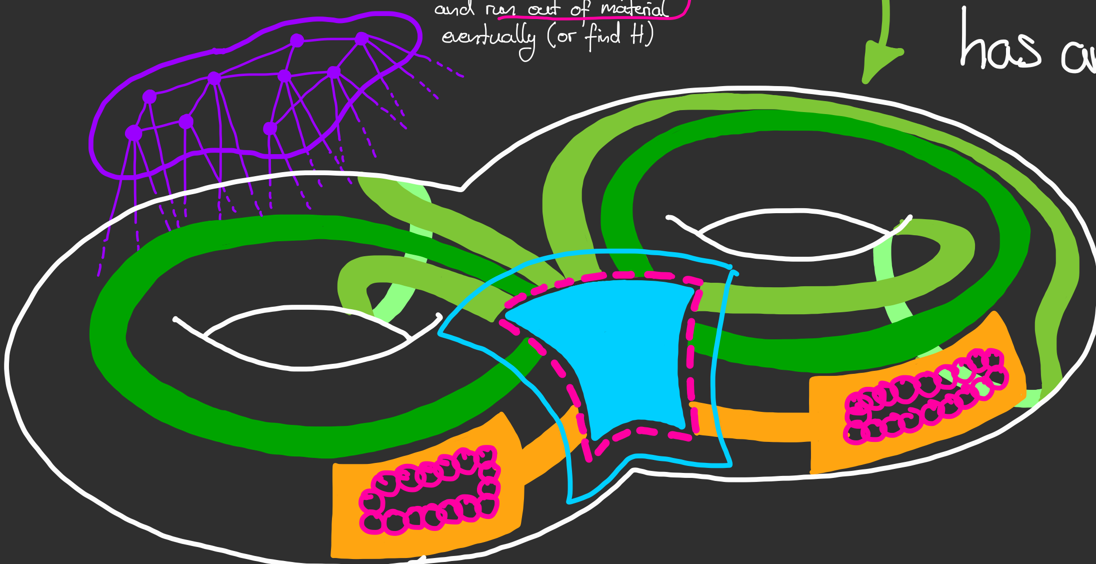

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H-minor-free graphs are tree-like! (GMST)

irrelevant vertices

↓

H-Minor Checking $\in$ P!

WQO!

minor-monotone problems $\in$ FPT [Fellows, Langston 1988]

Tracting the Intractable

treewidth, GMST, and efficient minor-testing

planar graphs [Demaine, Fomin, Hajiaghayi, Thilikos]

Baker's Techique (1994) Bidimensionality (2005)

layerings, partitions, contractions, Γ-phenomenon

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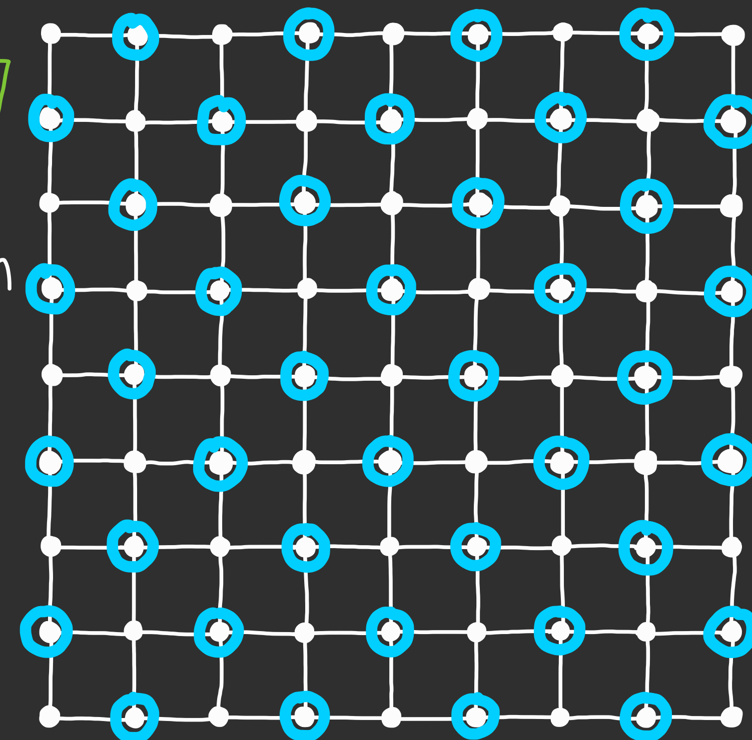

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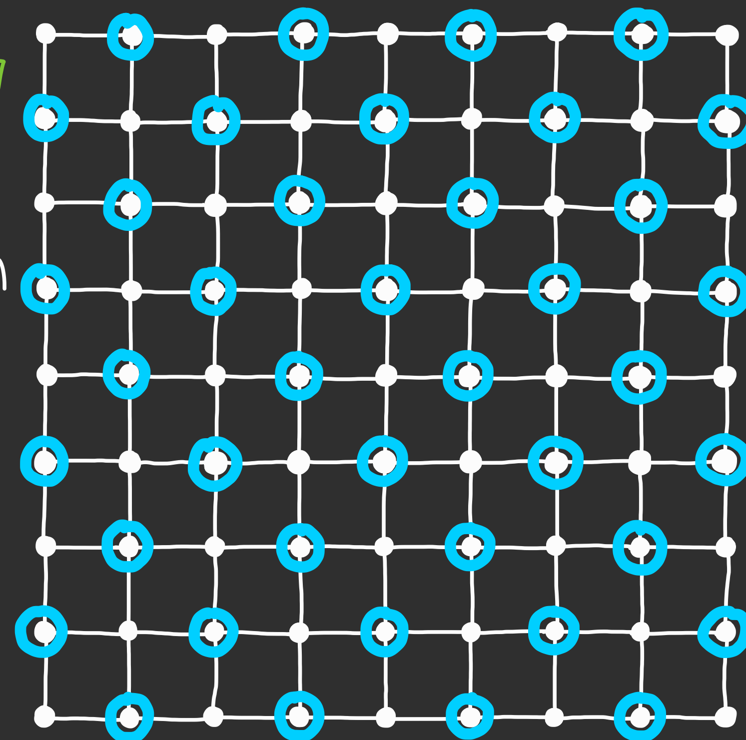

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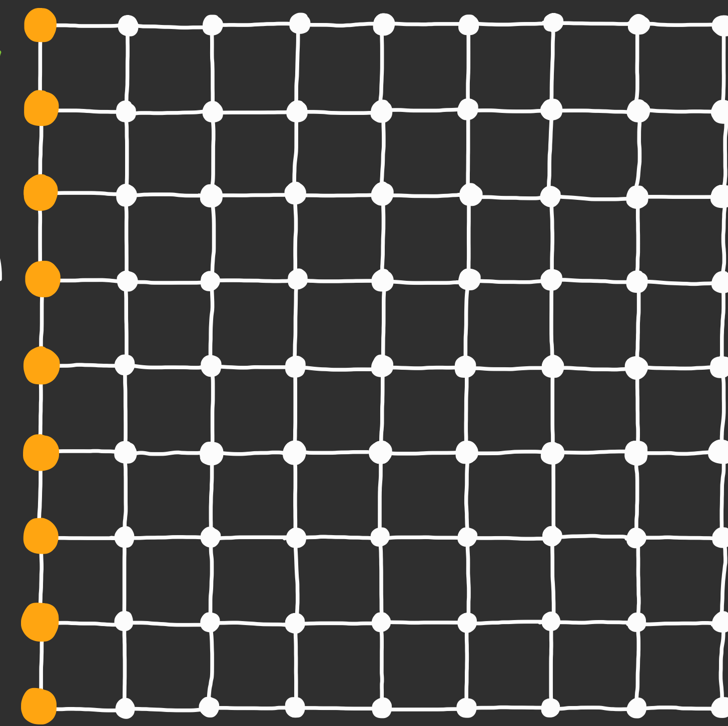

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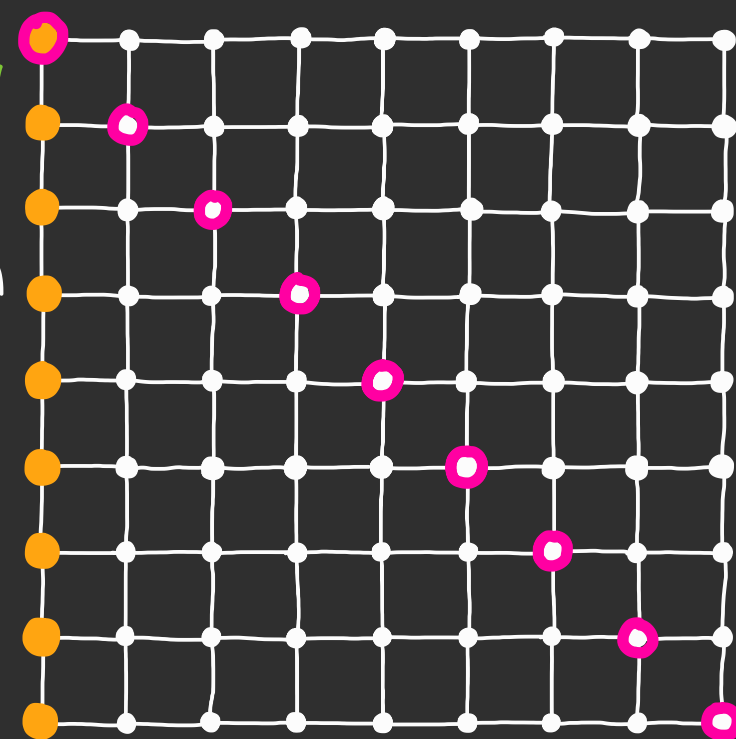

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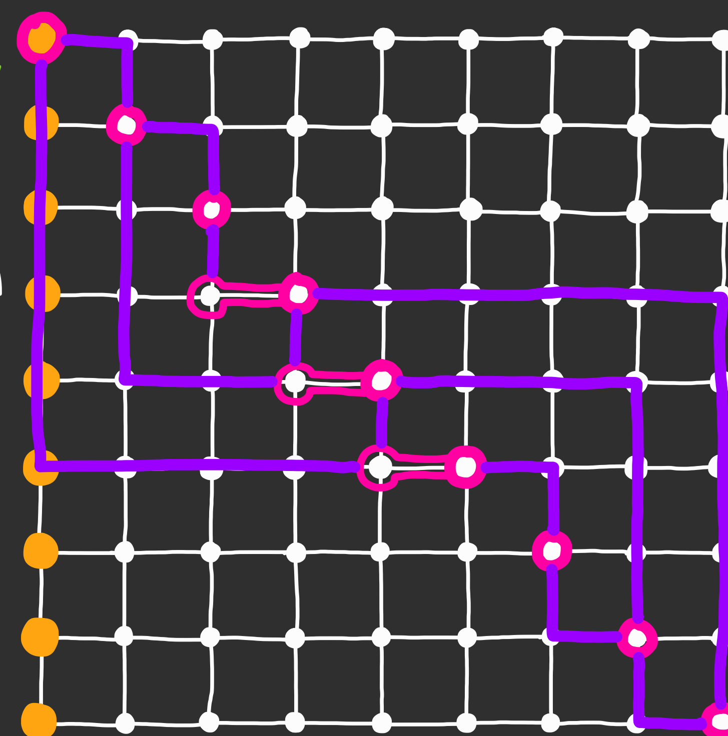

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replaces both, vortices and apices in the GMST

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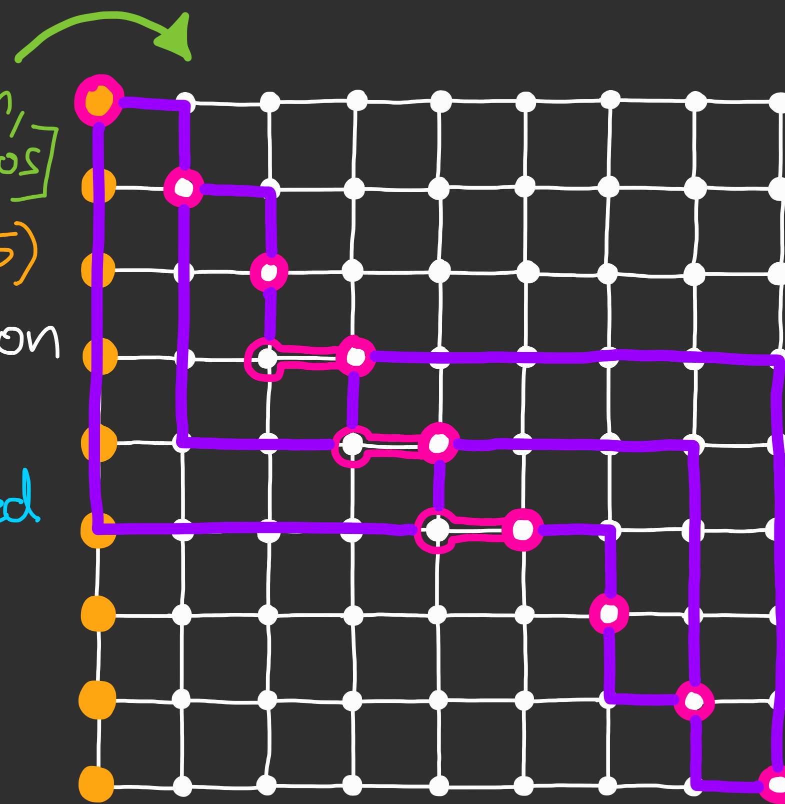
MSO$_2$-Model-Checking is in FPT for graphs of bounded treewidth [Courcelle 1990]



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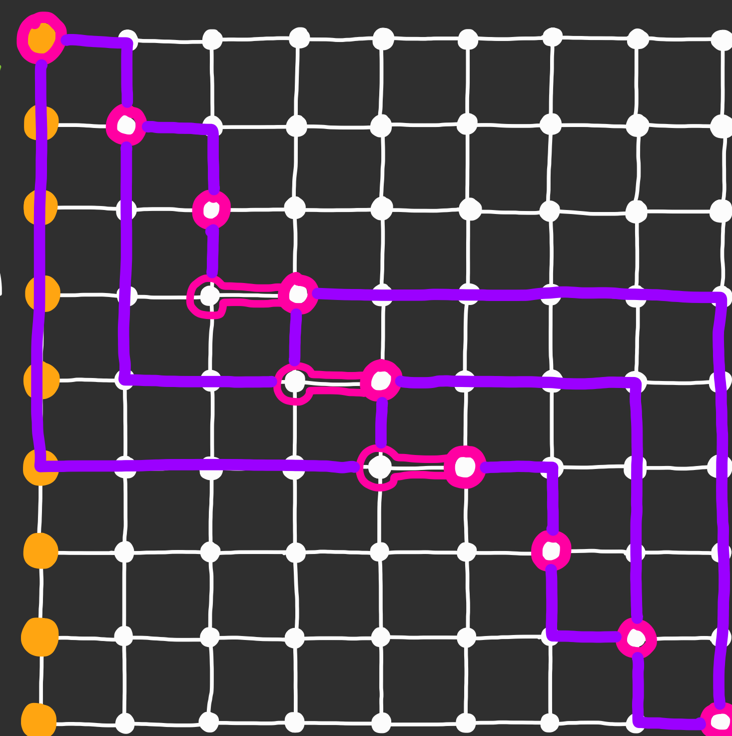

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$MSO/_{bidim}$ + dp - Model-Checking is in FPT for H-minor-free graphs

[Sau, Stamoulis, Thilikos 2025+]

Where to go from here?

Some "open" Problems

Conjecture [Hadwiger 1943]

$\chi(G) \leq t-1$ for all K_t-minor-free graphs

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Theorem [Kawarabayashi, Reed 2009]
There exist functions g, f s.t. for all t and graphs G, one of the following can be found in time $g(t) \cdot n^2$

Hadwiger's Conjecture is decidable

- a $(t-1)$-colouring for G
- a K_t-minor for G
- a minor H of G on $\leq f(t)$ vertices s.t. $\chi(H) \geq t$ and H is K_t-minor-free.

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Theorem [Norin & Thomas 2009^+]
There is a function f s.t. for all t, every t-connected K_t-minor-free graph with $\geq f(t)$ vertices is $(t-5)$-apex planar

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Structural & Computational Dichotomies

graphs

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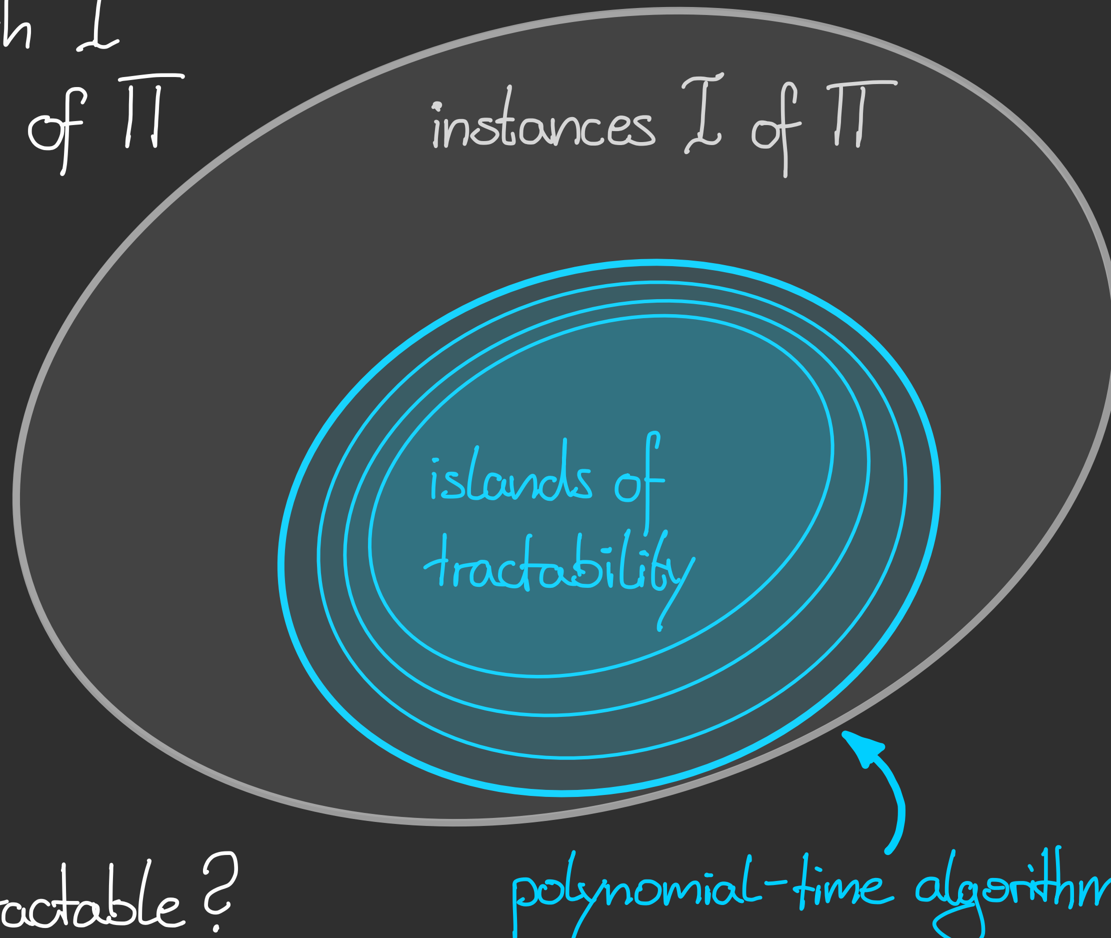

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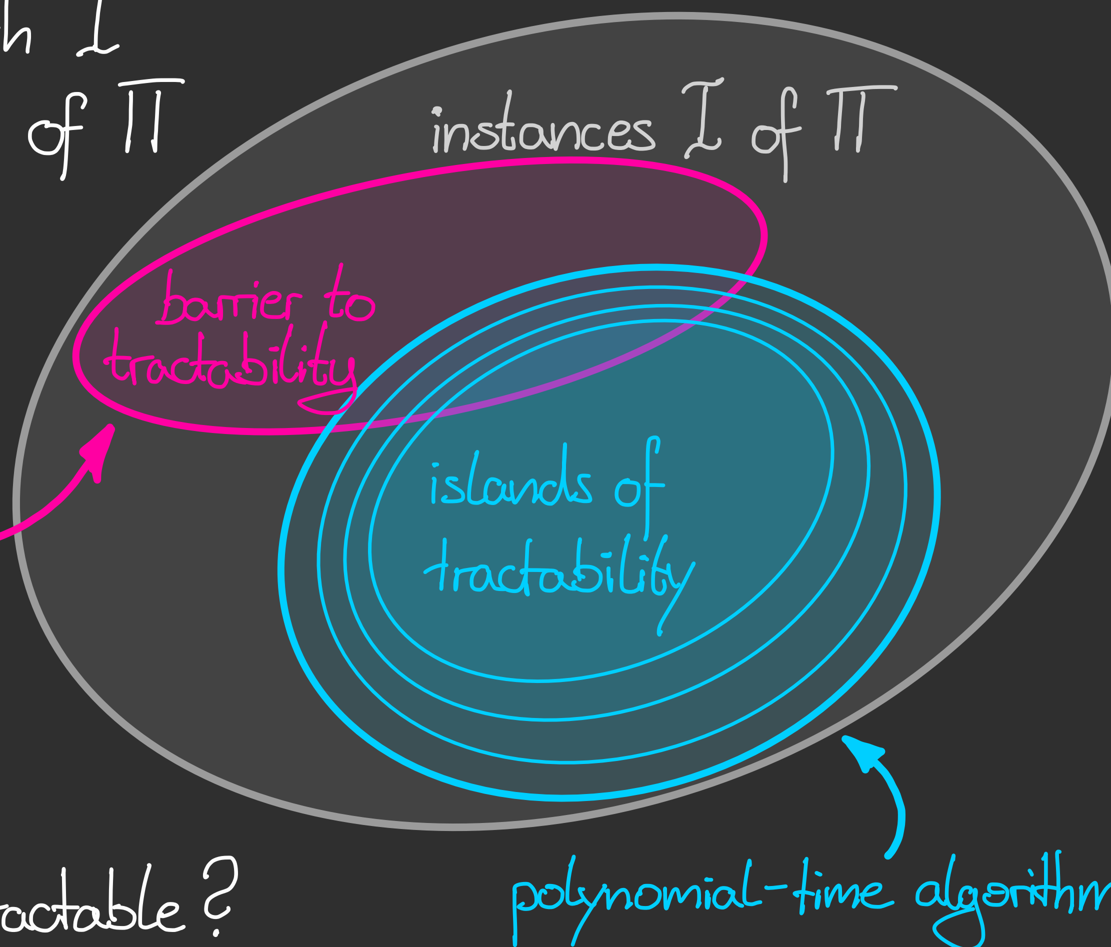

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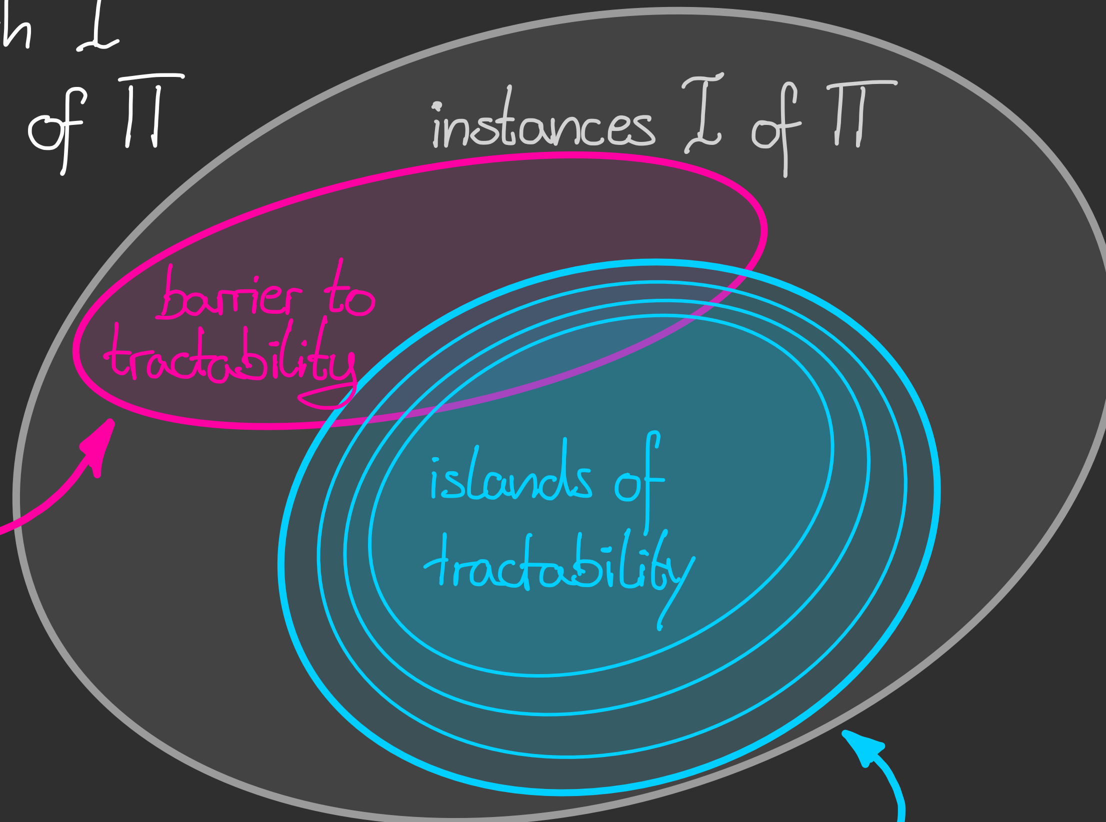


Π is hard on this class

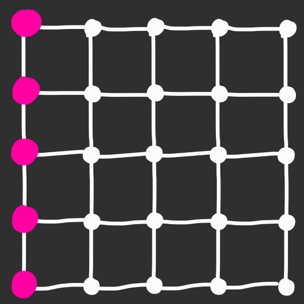
polynomial-time algorithm

Three Questions

1) For which $\leq$-closed classes is Π tractable?
2) What are the $\leq$-minimal barriers for Π?
3) Can we obtain a full dichotomy for Π under $\leq$?

capture the barriers with a $\leq$-monotone **graph parameter**

A Structure Theorem for Bidimensionality



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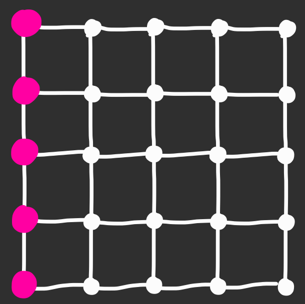

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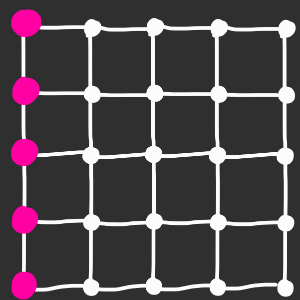

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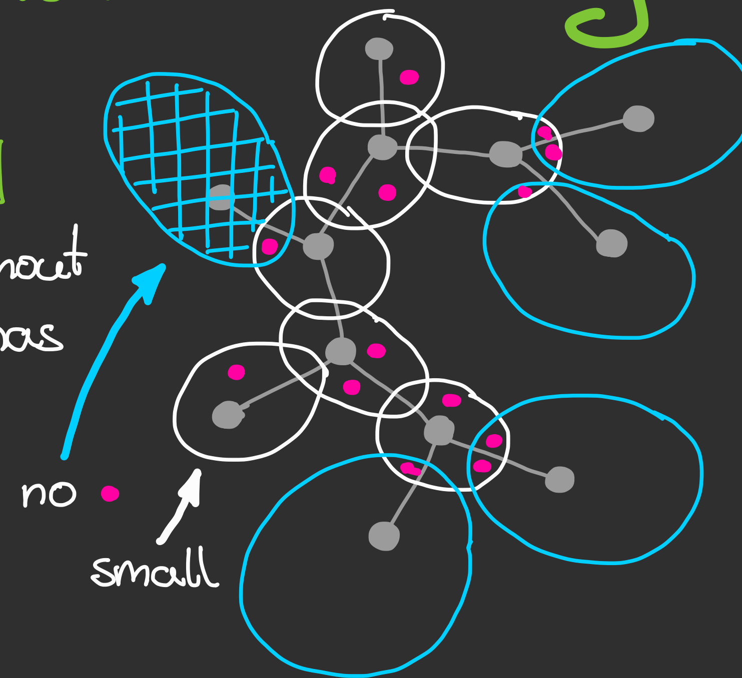

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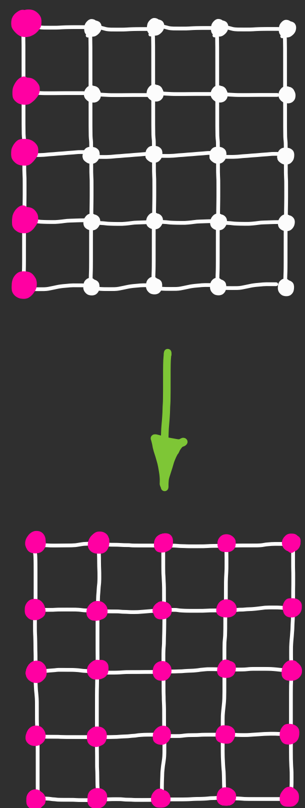

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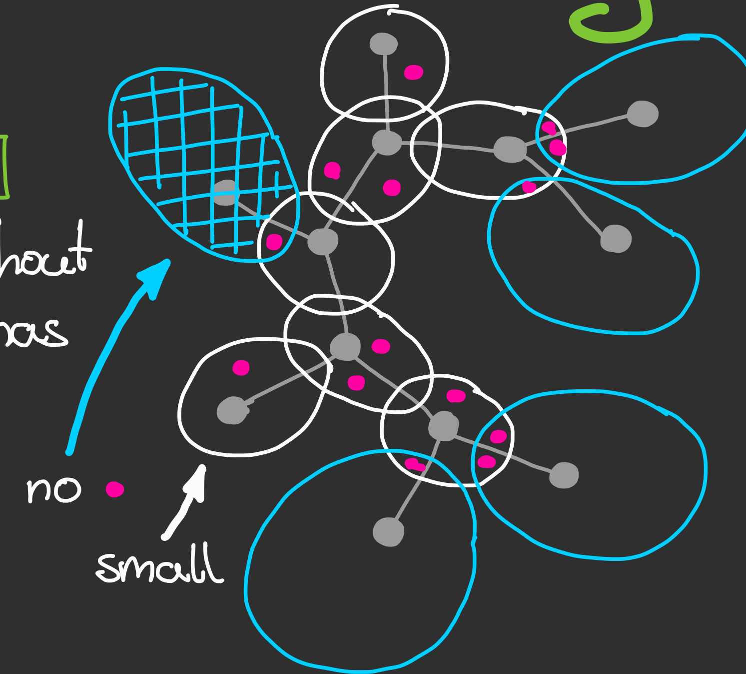

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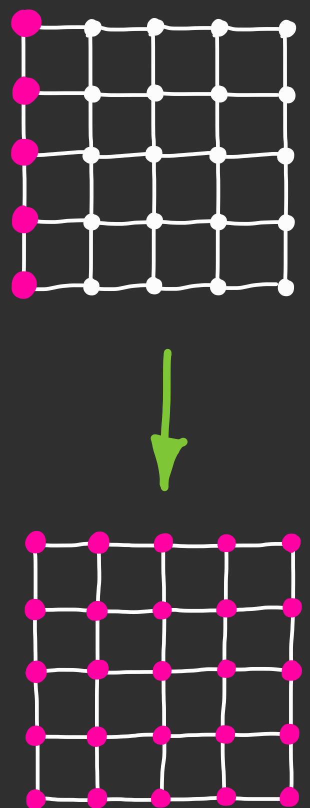

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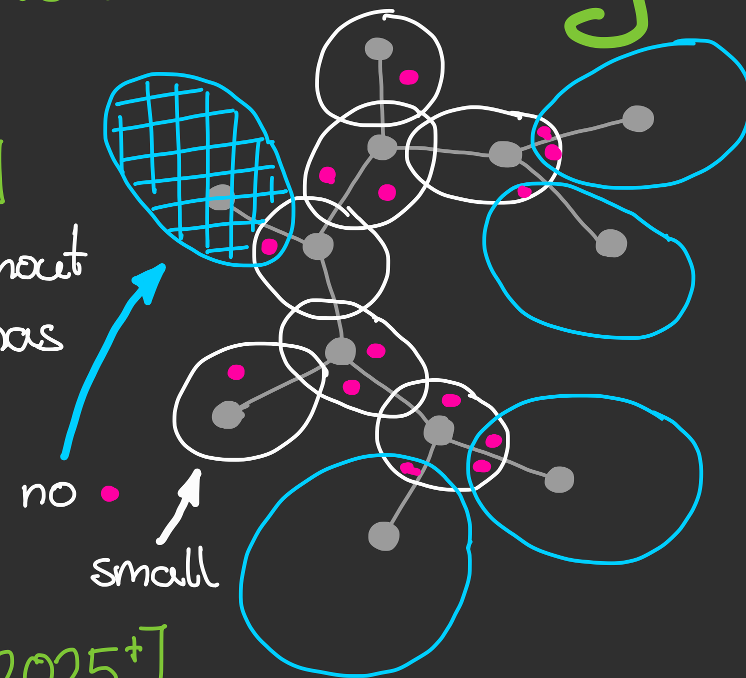

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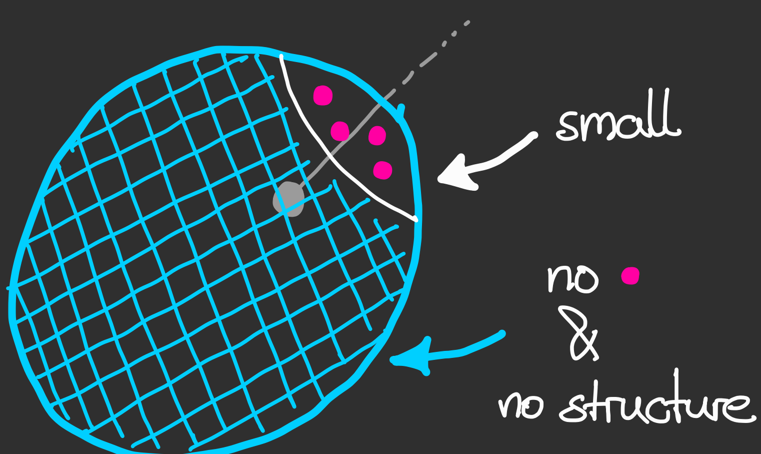

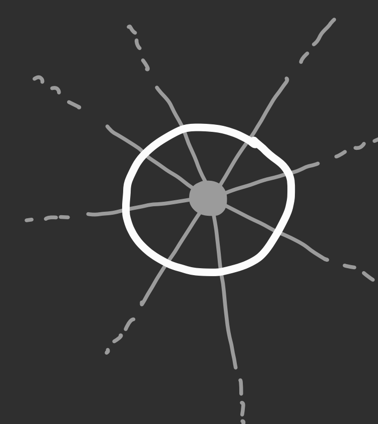

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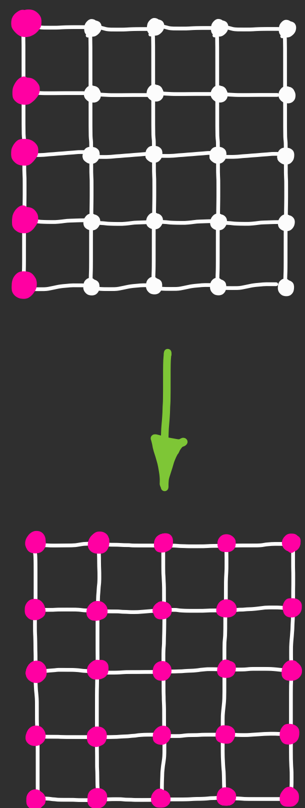

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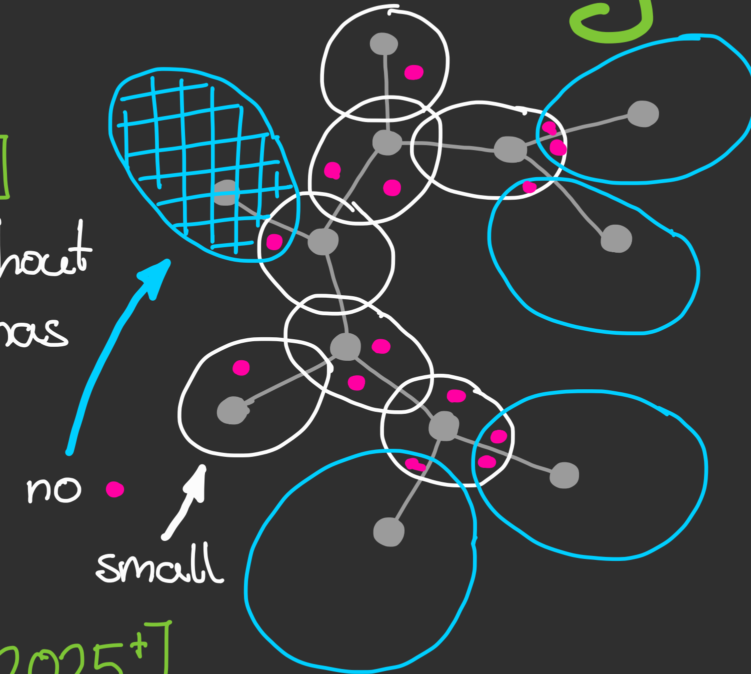

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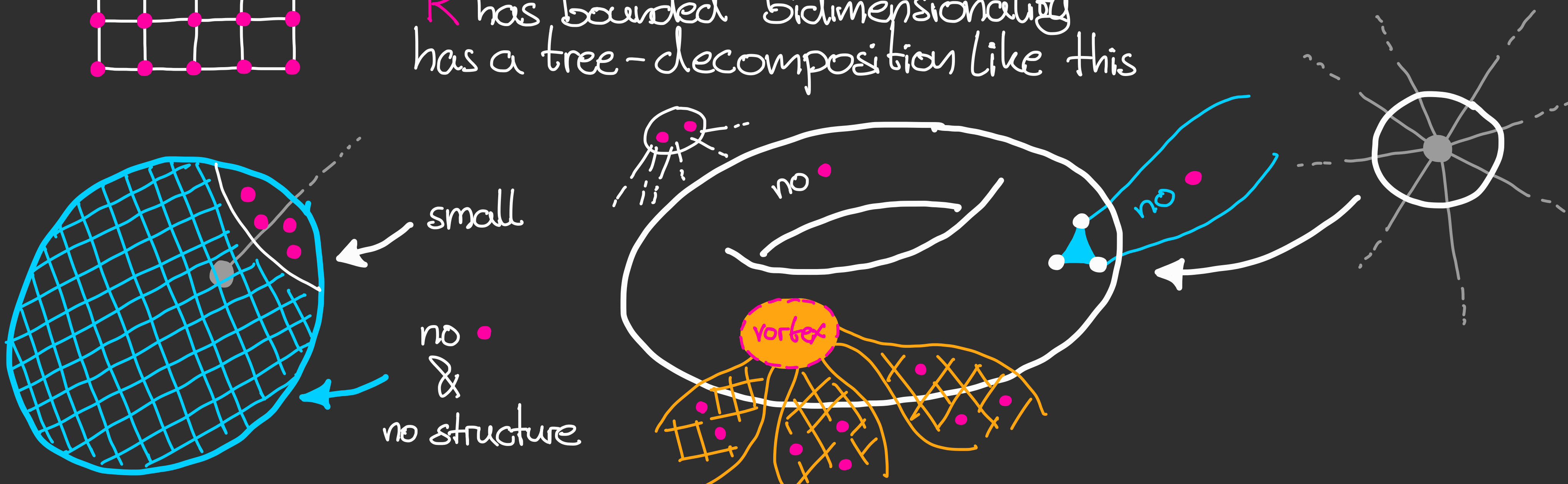
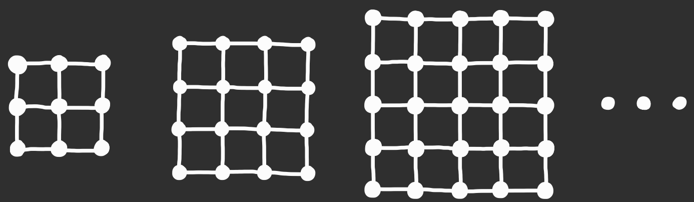


Universal Obstructions



$\mathcal{H} = \{H_i\}_{i \in \mathbb{N}}$ parametric graph

$H_1 \leqslant H_2 \leqslant H_3 \leqslant \dots$

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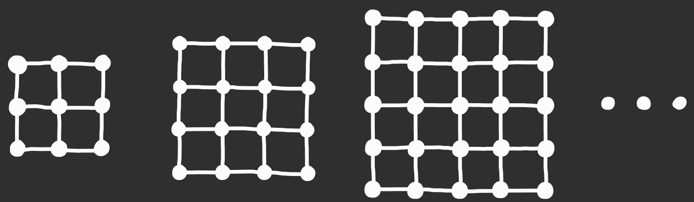

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treewidth

p graph parameter (minor monotone)

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{grids}

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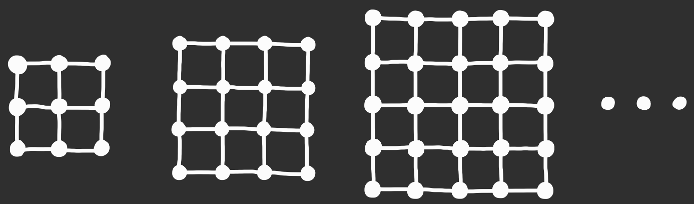

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Every minor-monotone graph parameter has a finite universal obstruction.

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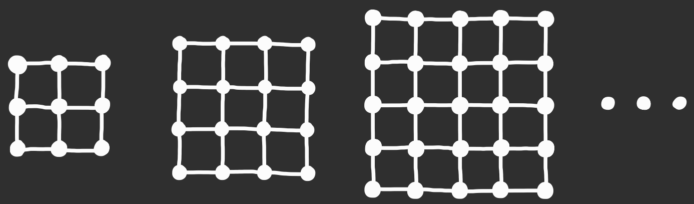

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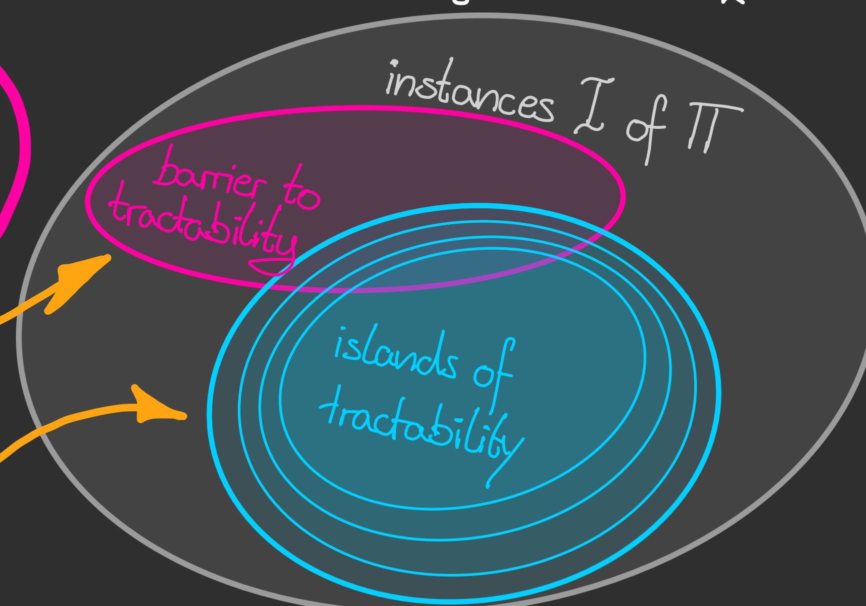
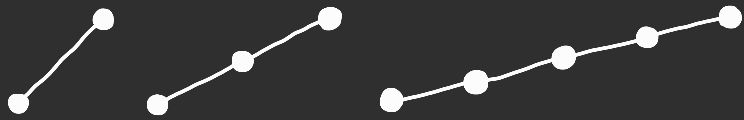


p is bounded

Milestones



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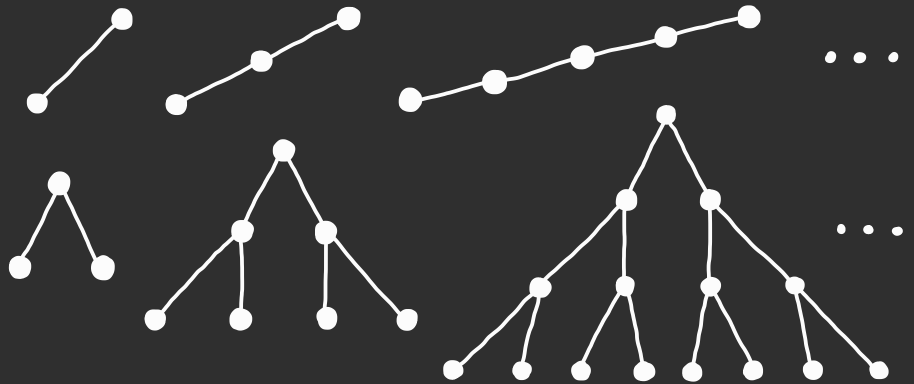

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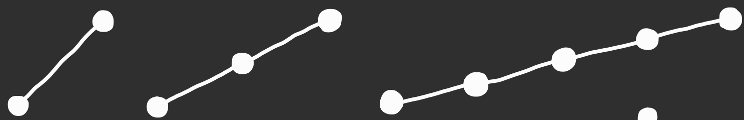

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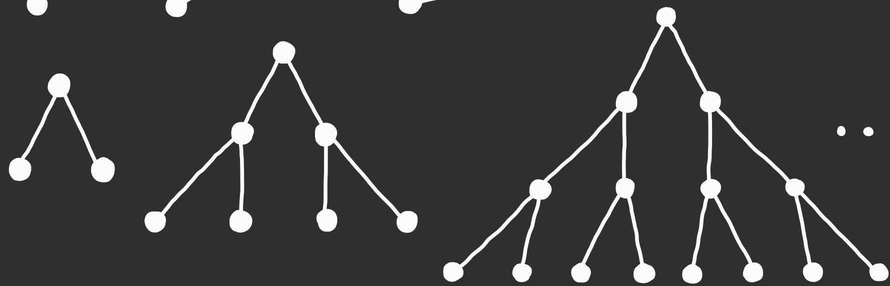

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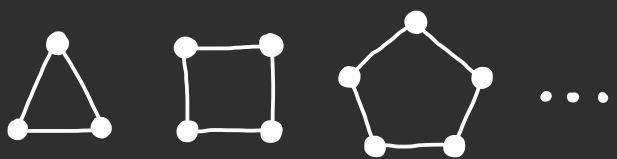

“K_3-hitting set”

… → feedback vertex number [Erdős, Pósa 1968]

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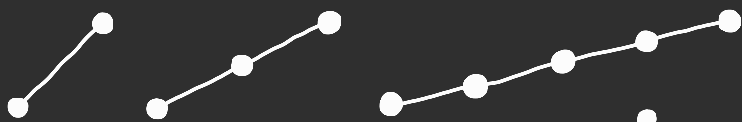

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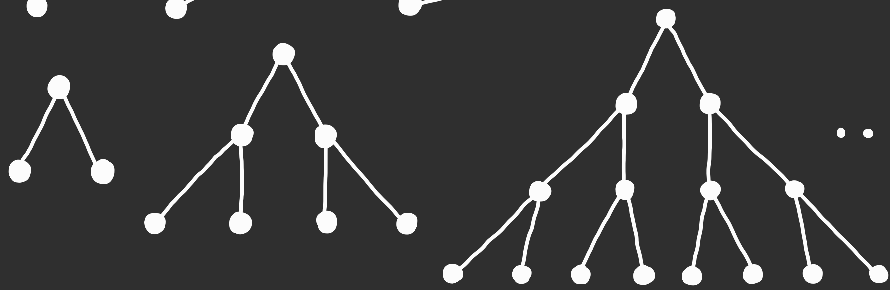

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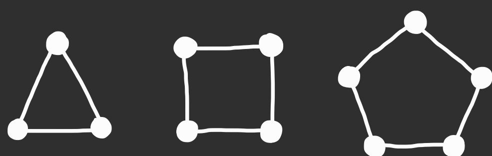

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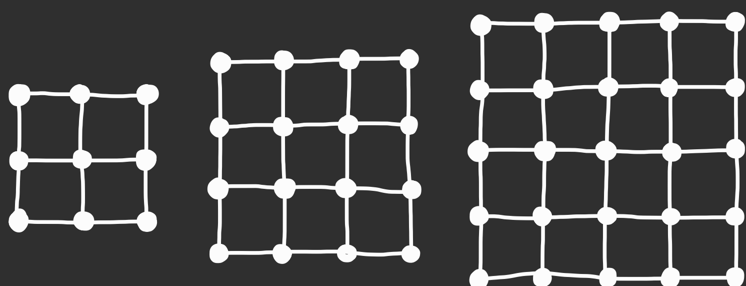

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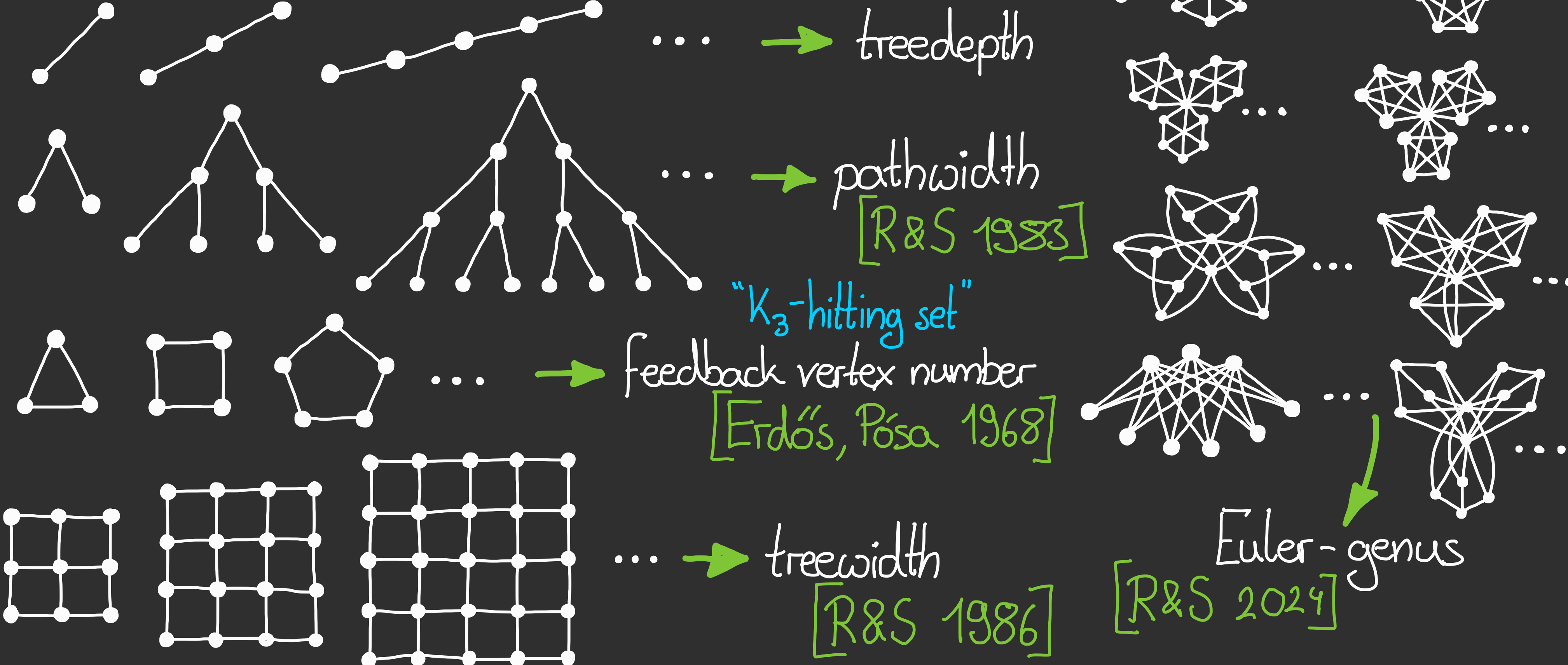
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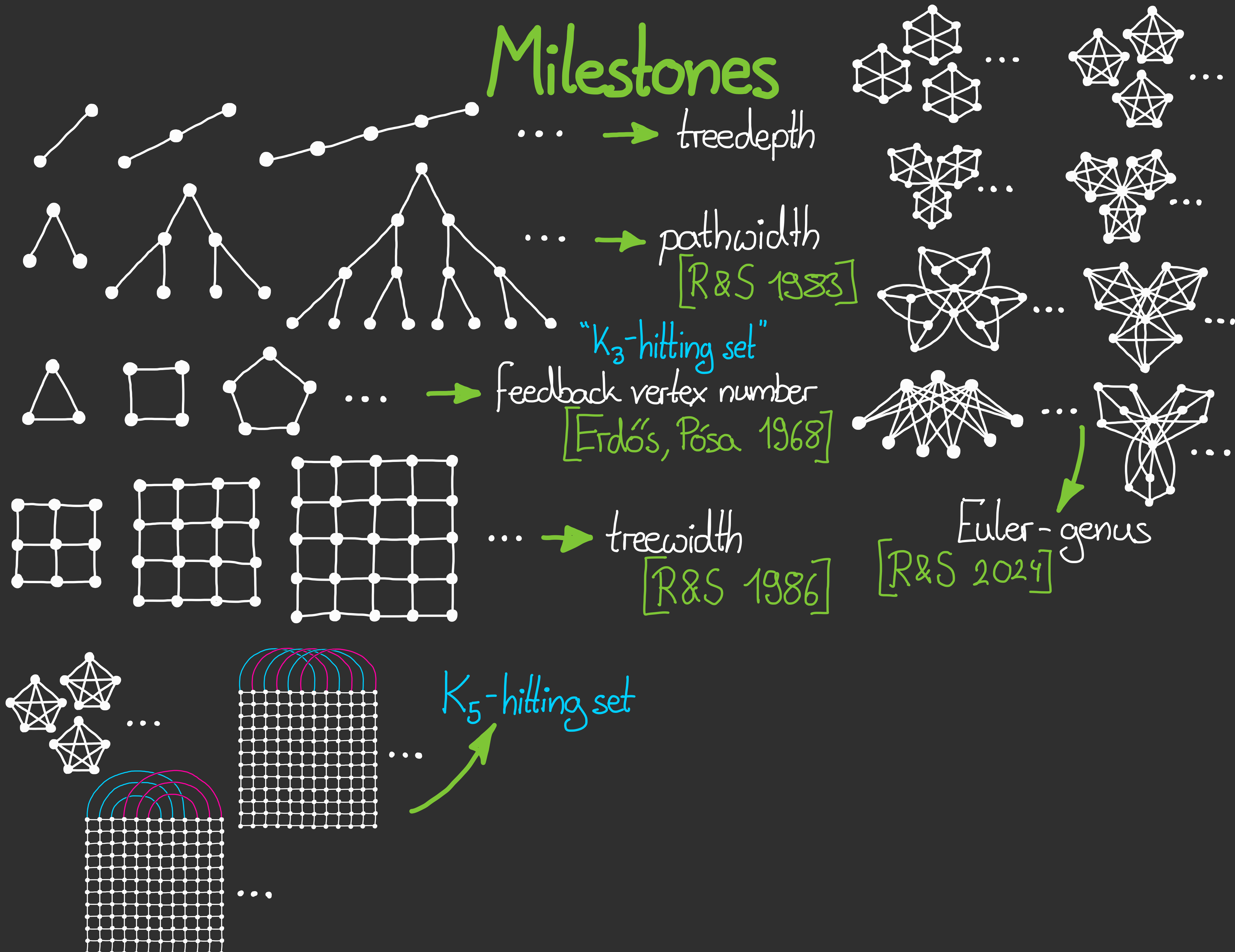
Milestones



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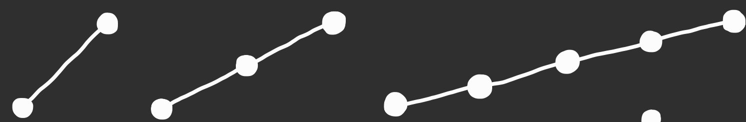
K_5-hitting set

Milestones



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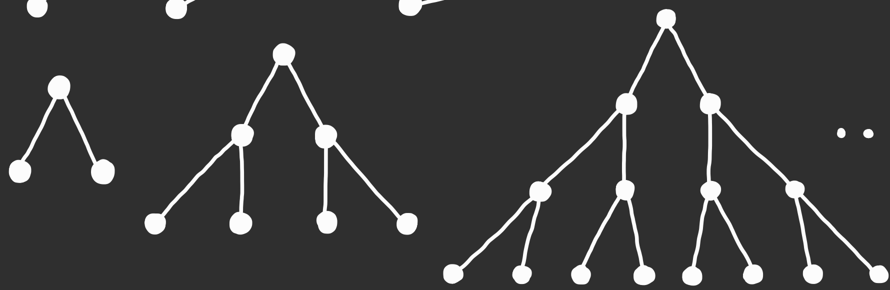

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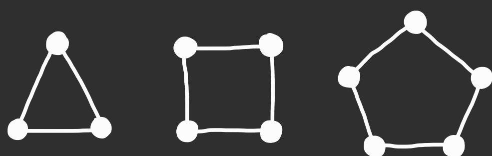

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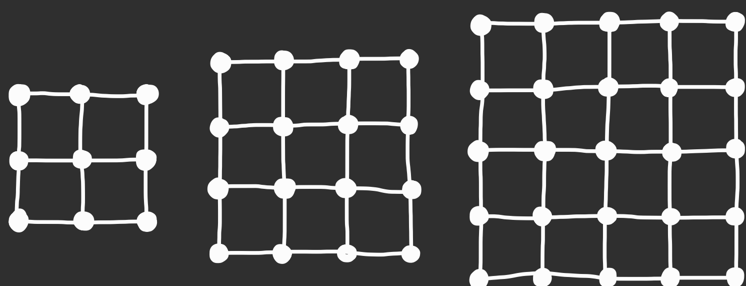

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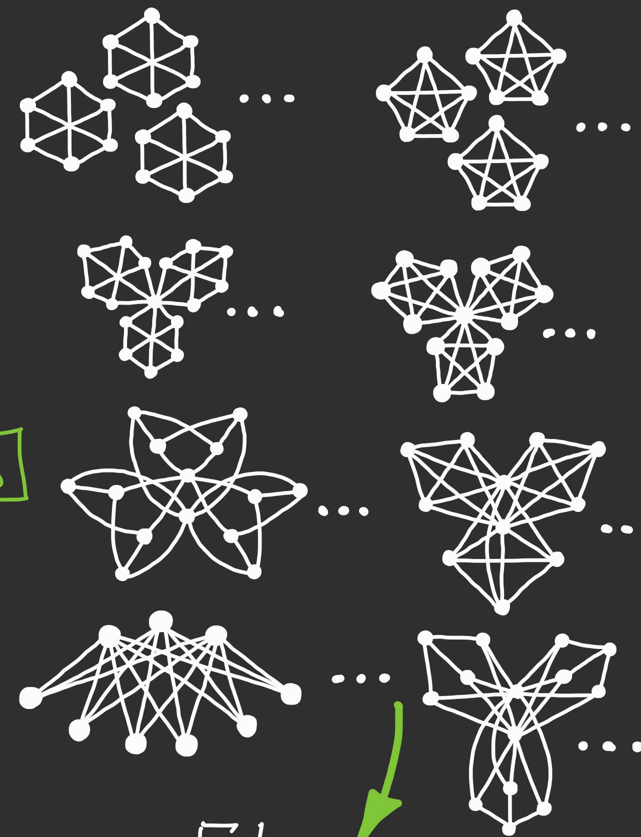

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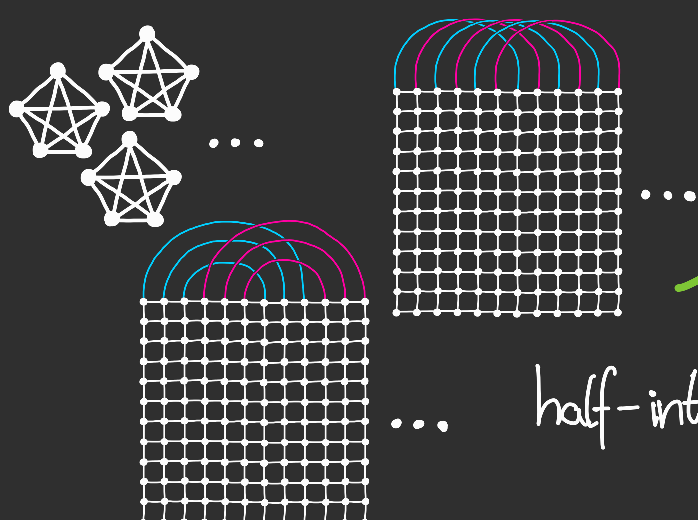
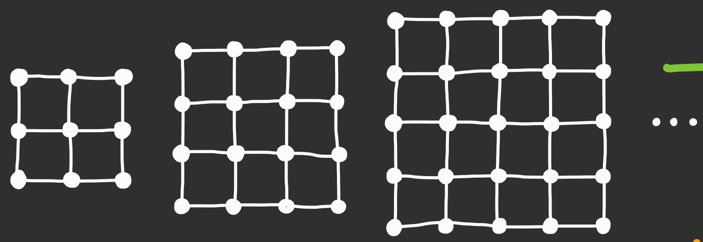


K_5-hitting set

For every graph H, H-hitting set has a finite universal obstruction.
[Paul, Protopapas, Thilikos, W. 2024+]

half-integral Erdős–Pósa property for all graphs
[Liu 2022]

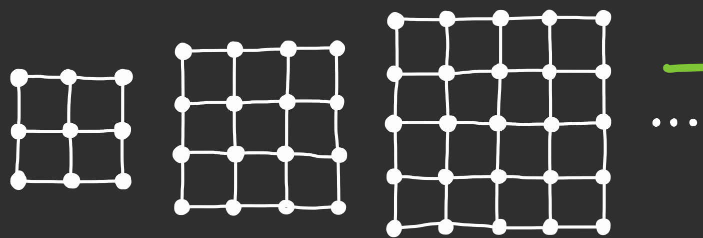
Omnivores



any parametric graph gives rise to a class

minor-closure = planar graphs

Omnivores



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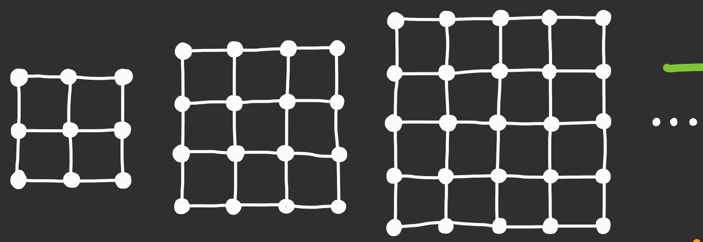
Omnivores



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Lemma [Paul, Protopapas, Thilikos 2024+]
A minor-closed graph class has an omnivore if and only if it is an ideal

$\downarrow$-closed
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$H \preceq G$
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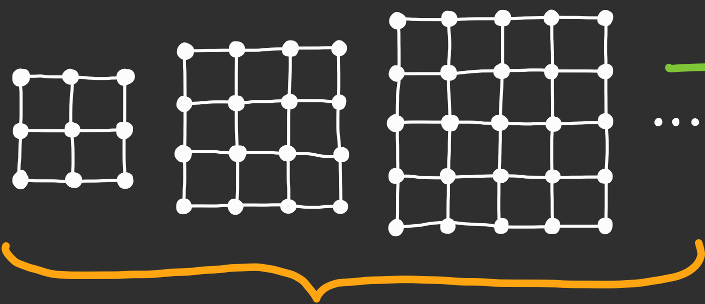

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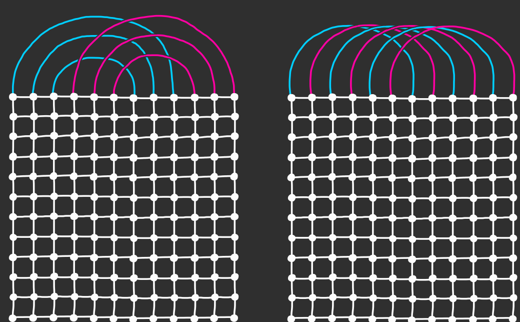

toroidal graphs $\cup$ projective graphs

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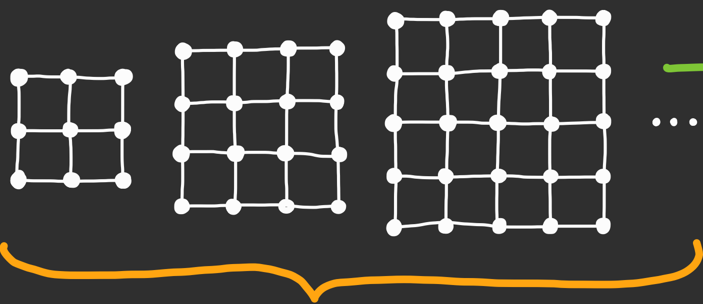
every minor-closed class is generated by finitely many parametric graphs

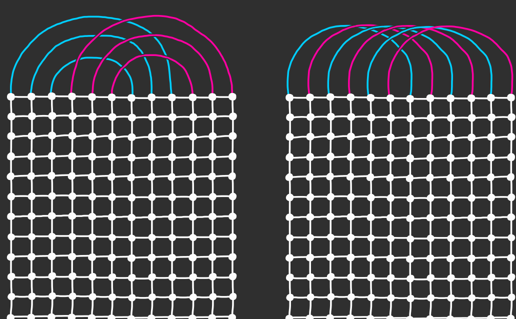
Omnivores



minor-closure = planar graphs

any parametric graph gives rise to a class

Let $\mathcal{C}$ be a minor-closed graph class. A parametric graph $\mathcal{H}$ is an omnivore for $\mathcal{C}$ if $\mathcal{C} = \text{closure}(\mathcal{H})$.



toroidal graphs $\cup$ projective graphs

Lemma [Paul, Protopapas, Thilikos 2024[+]]
A minor-closed graph class has an omnivore if and only if it is an ideal

$\downarrow$-closed
+
$H \preceq G$, $J \preceq G$

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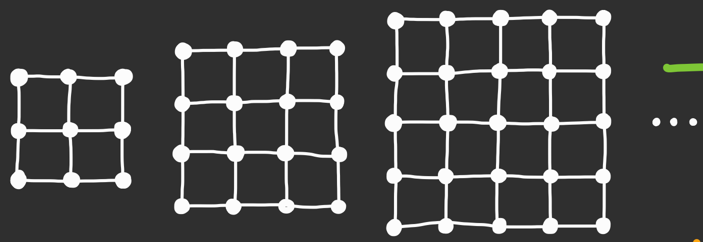
Challenge provide efficient omnivores for your favourite minor ideals

Omnivores



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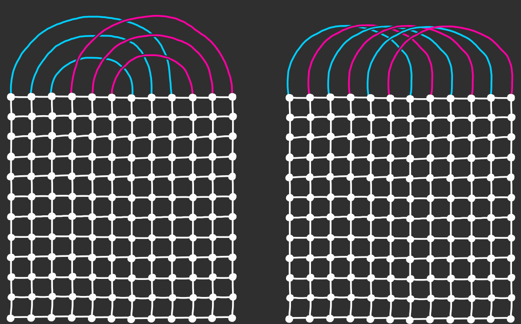

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Beyond WQO

Let p be a minor-monotone parameter.

$$\mathbb{C}_p := \{\mathcal{C} \text{ minor-closed} : p \text{ is bounded on } \mathcal{C}\}$$

$\subseteq$-closed
class of classes

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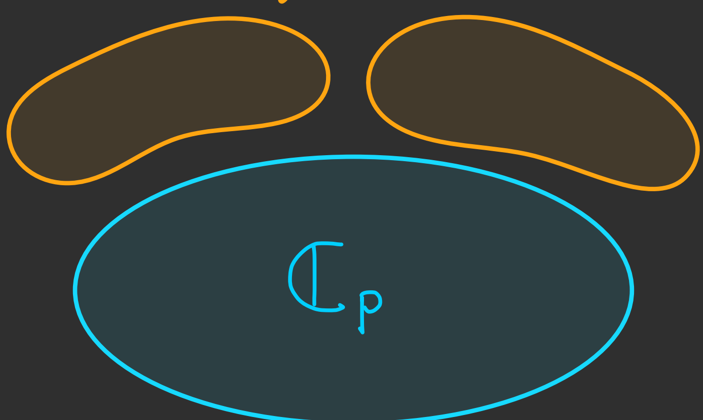

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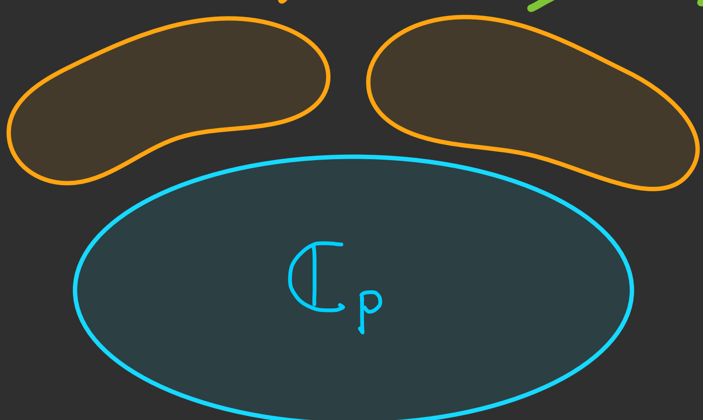

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$\subsetneq$-closed class of classes

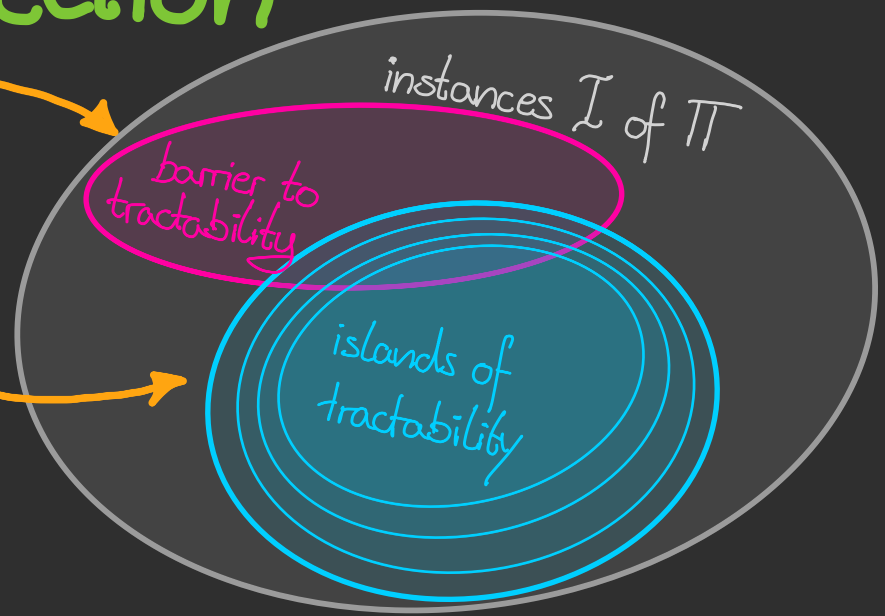
Conjecture

Minor-closed classes are WQO under $\subsetneq$.

Problem Collection

class of all minors
of universal obstruction

p is bounded



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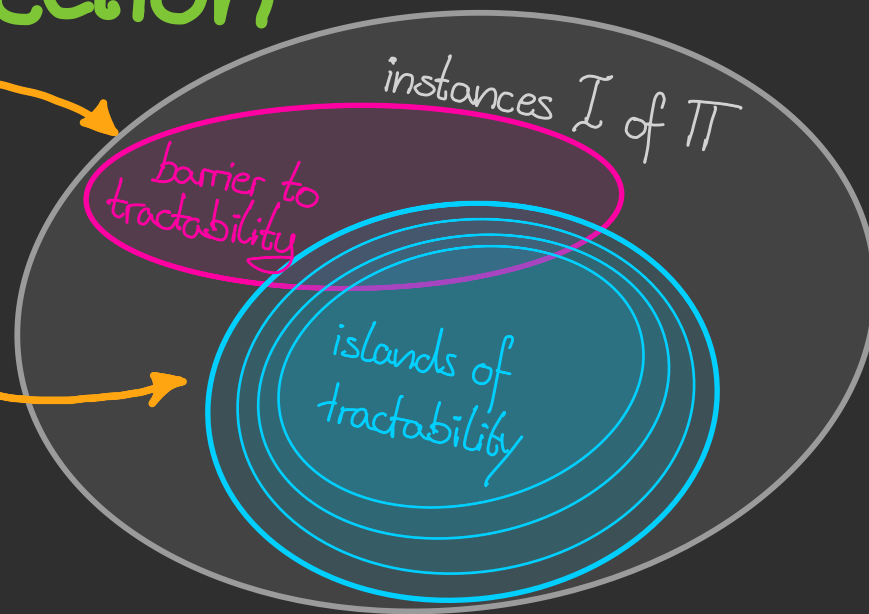

Maximum Cut

↳ **NP**-hard on apex-planar graphs
[Barahona 1983]

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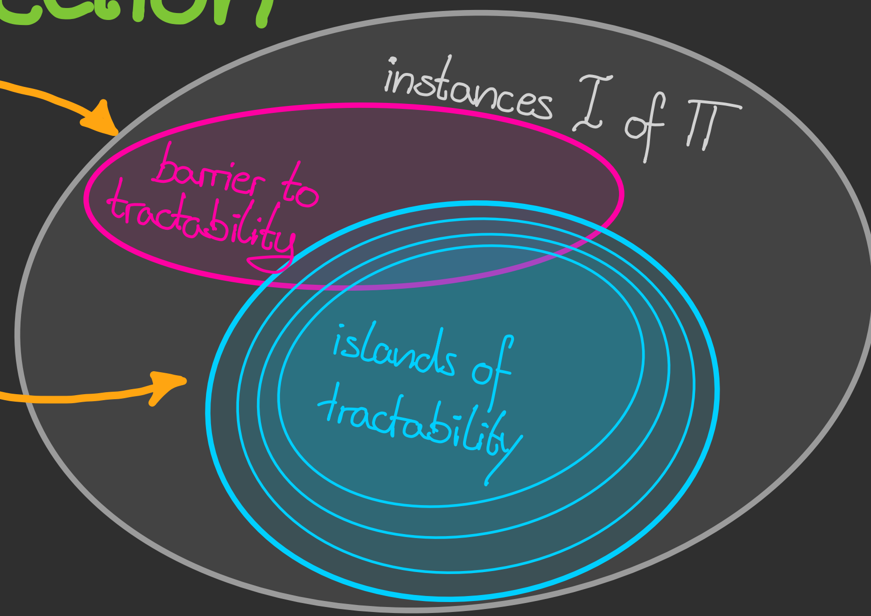

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For every apex-planar graph H, MAXCUT can be solved in polynomial time on H-minor-free graphs.

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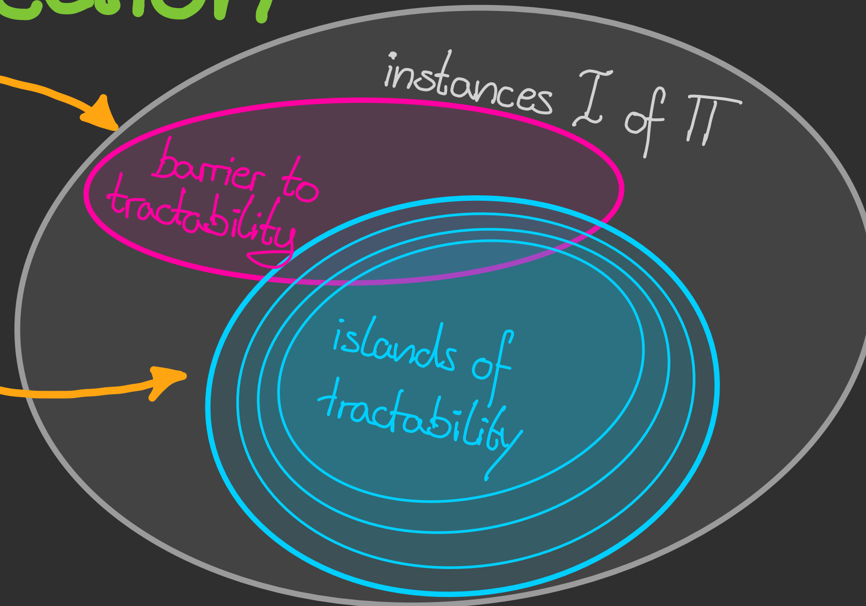

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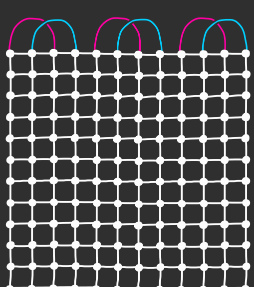

How to deal with vortices?

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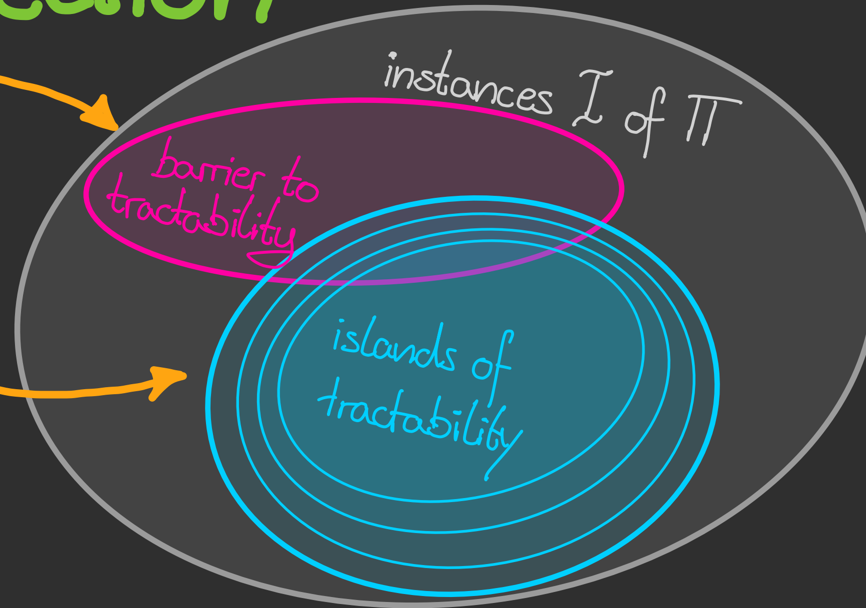

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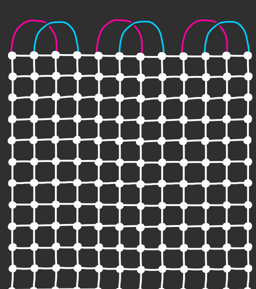

How to deal with vertices?

⋮

do you know other good candidates?

Problem Collection

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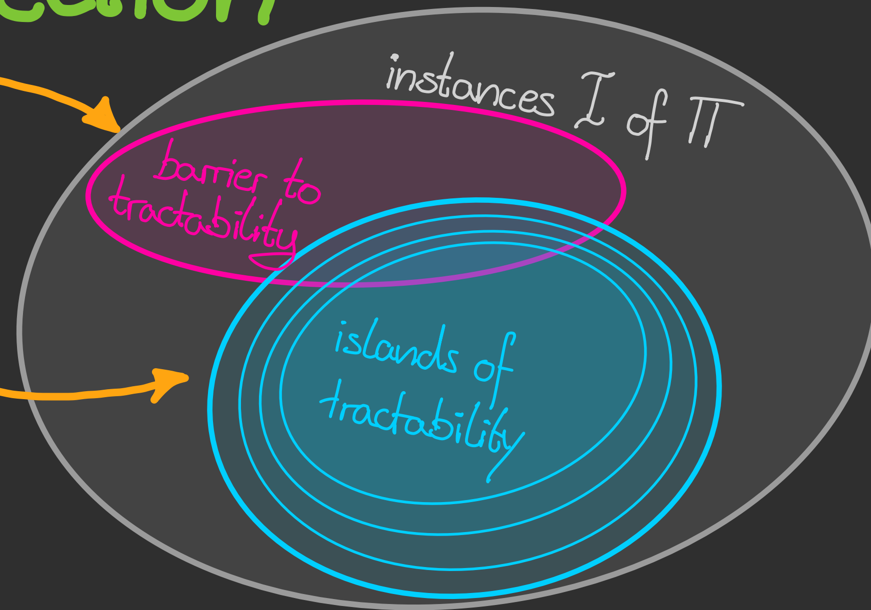

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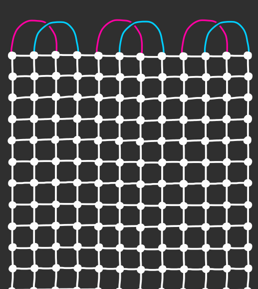

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- STEINER TREE
- R-SPANNING CYCLE

⋮

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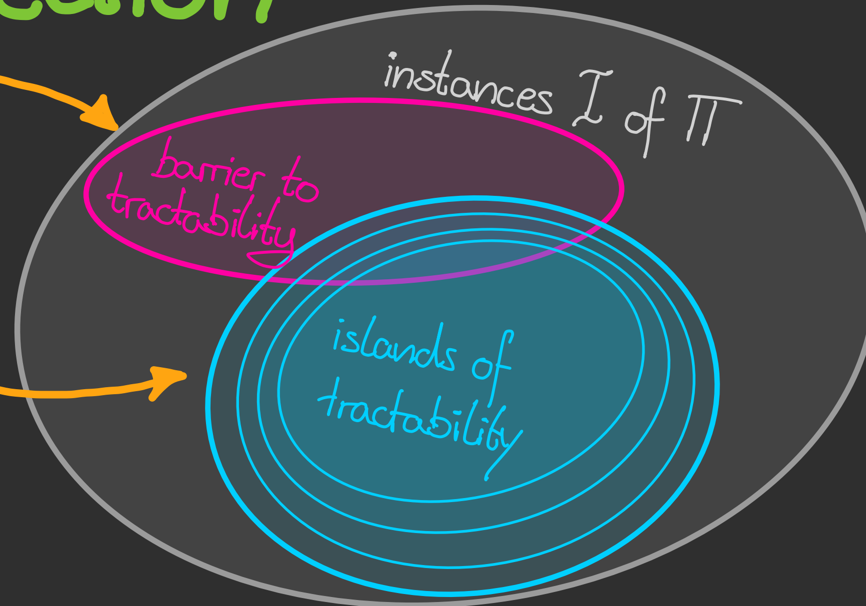

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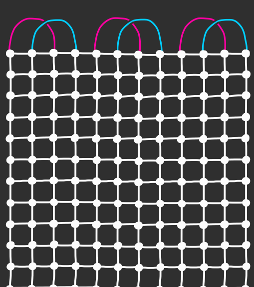

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Thank You!