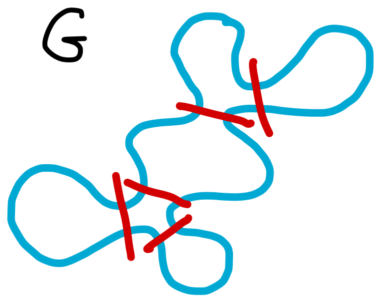
Cyclewidth and the Grid Theorem for Perfect Matching Width of Bipartite Graphs

WG 2019

Meike Hatzel   Roman Rabinovich   Sebastian Wiederrecht

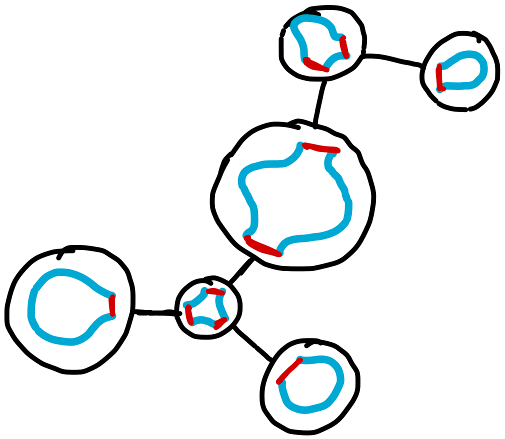
The Grid Theorem

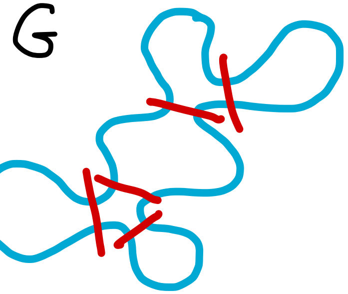


Goal: Decompose G
into a tree-like structure

The Grid Theorem

Treewidth





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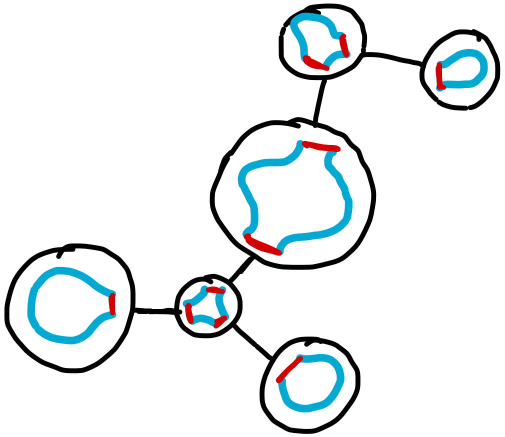
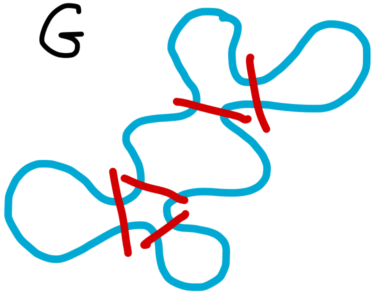


G



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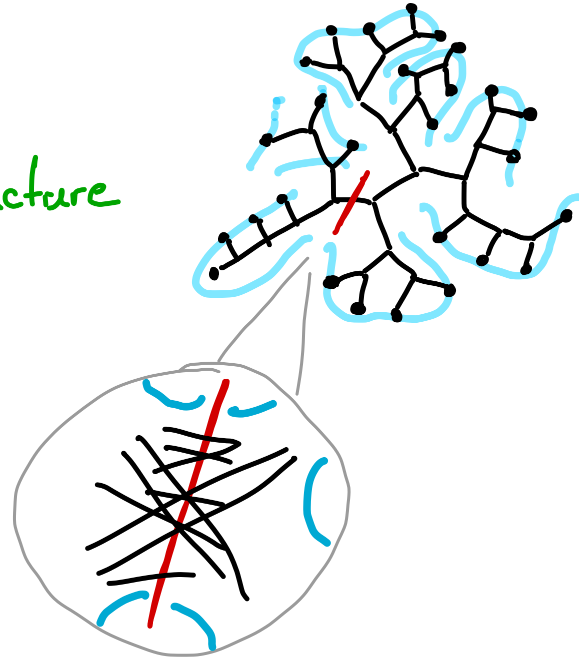
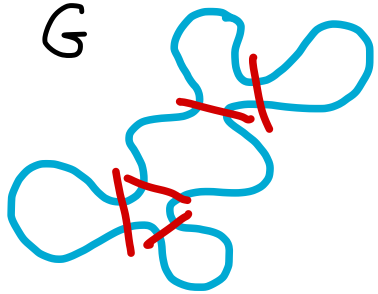
Maximum Matching Width



The Grid Theorem

G



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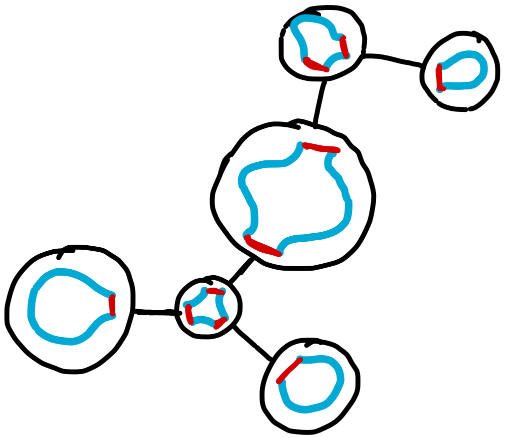
Treewidth



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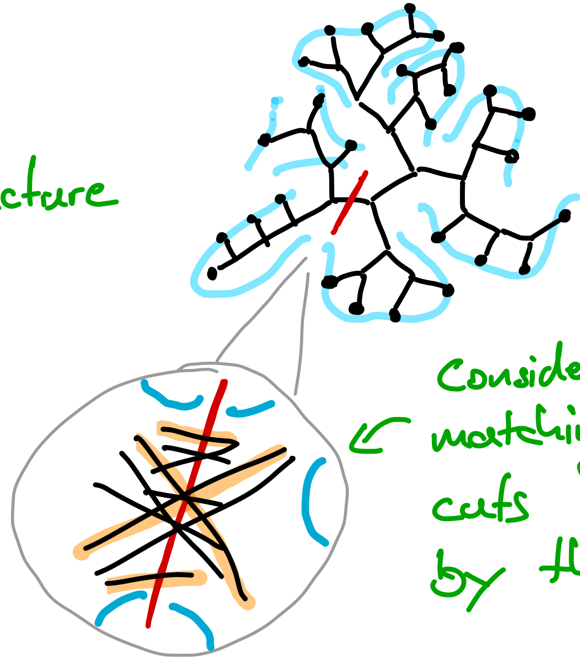
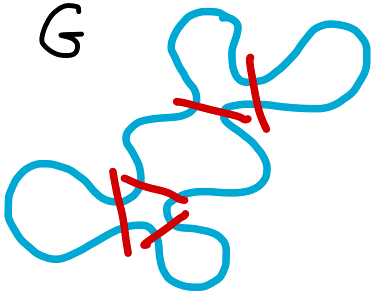


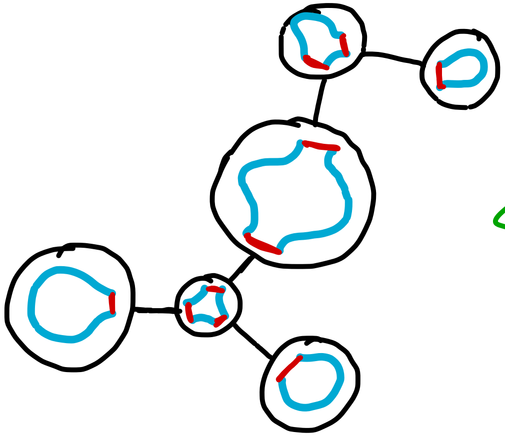
consider maximum matchings in edge cuts defined by the decomp

The Grid Theorem

G



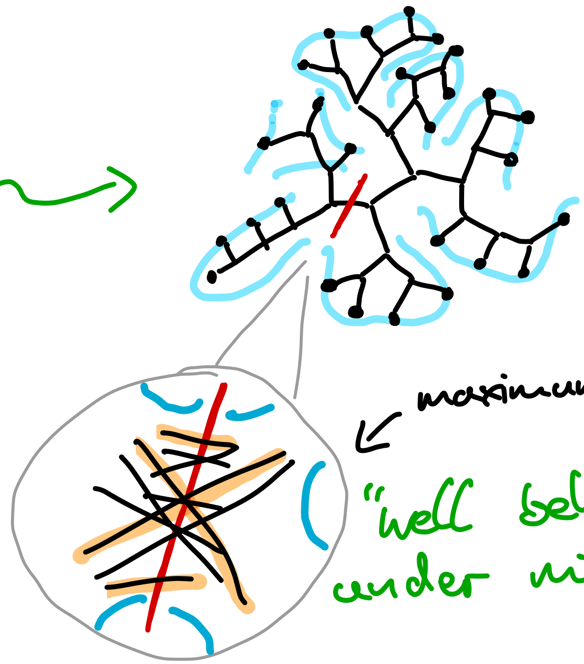
Treewidth



closed under minors

within a constant factor

Maximum Matching Width

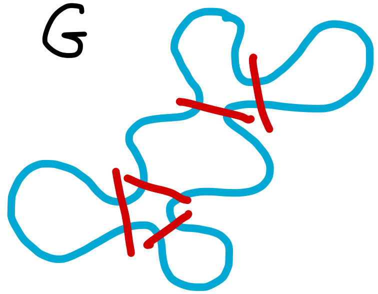


maximum matching

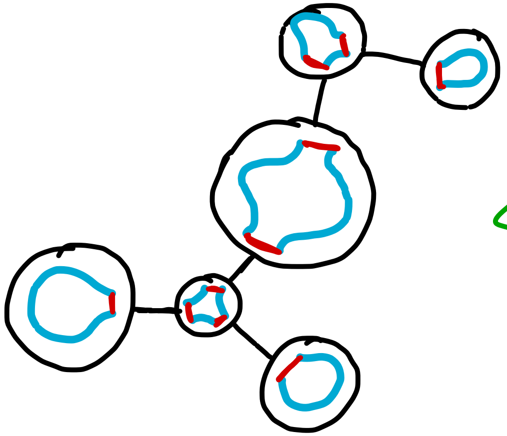
"well behaved" under minors

The Grid Theorem

G



Treewidth



closed under minors

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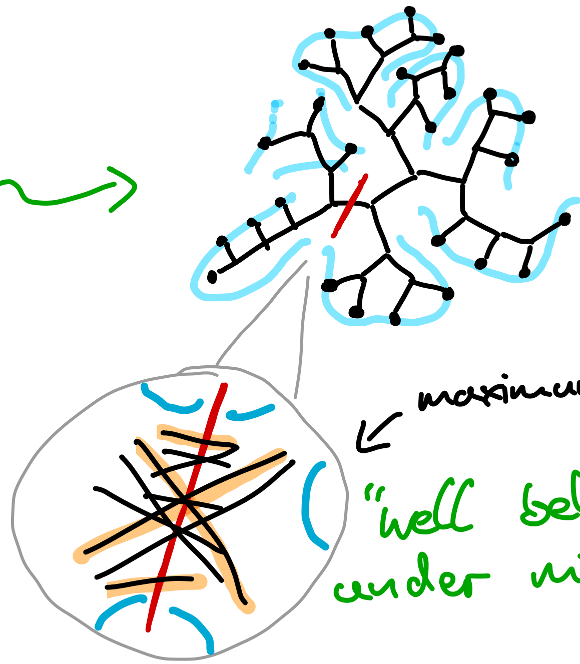
Maximum Matching Width

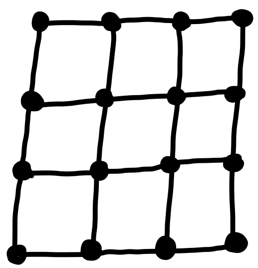


maximum matching

"well behaved" under minors

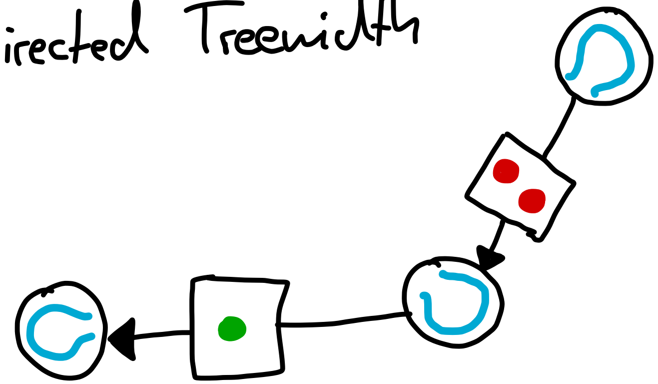
Theorem (Robertson & Seymour)

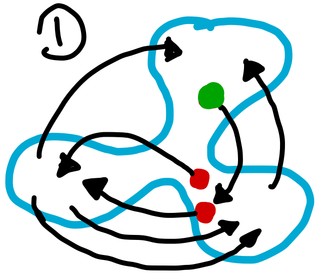
Either G has small treewidth or a large grid minor.



The Directed Grid Theorem

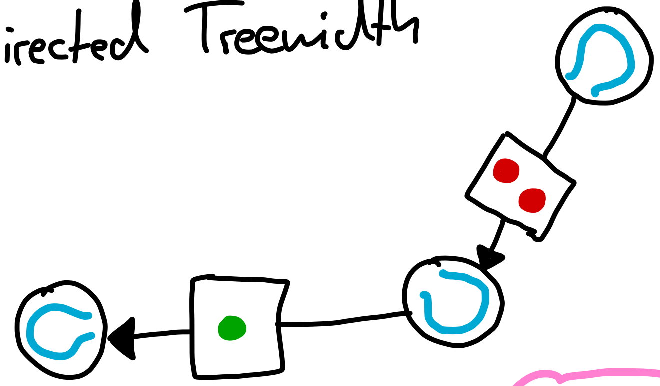
Directed Treewidth

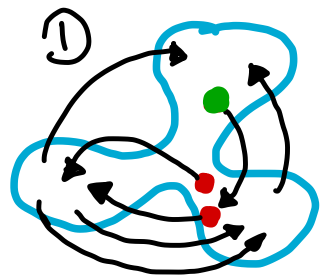




The Directed Grid Theorem

Directed Treewidth



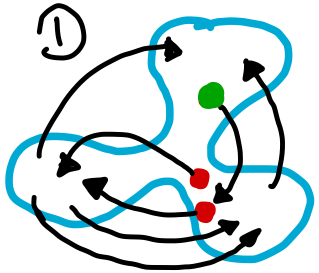


"well behaved" under butterfly minors

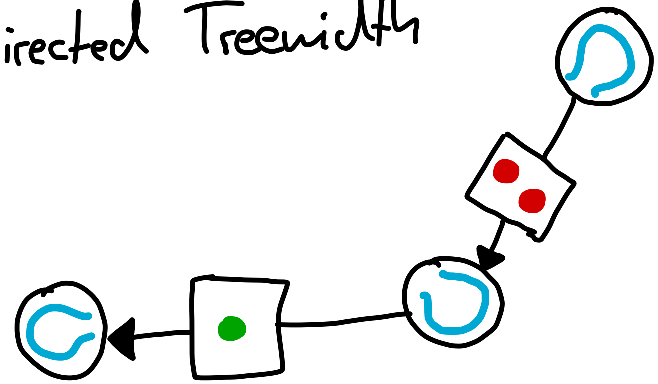
a directed minor notion where only "special" edges are allowed to be contracted

The Directed Grid Theorem

D



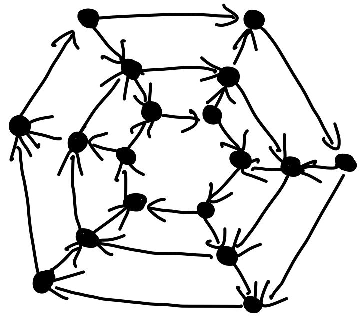
Directed Treewidth



"well behaved" under butterfly minors

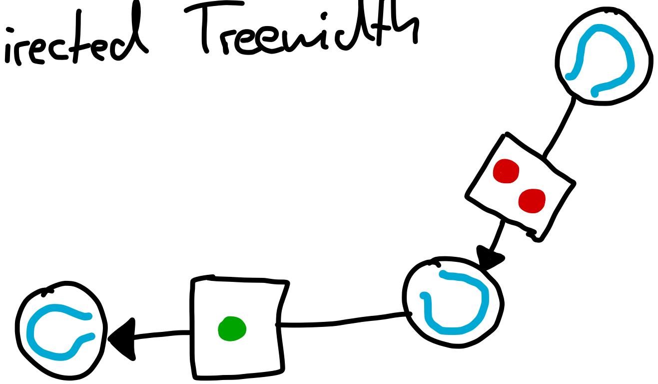
Theorem (Kawarabayashi & Kreutzer)

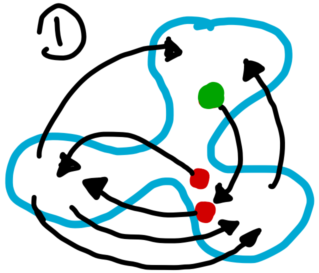
Either D has small directed treewidth or a large cylindrical grid as a butterfly minor.



The Directed Grid Theorem

Directed Treewidth





Will be filled later.

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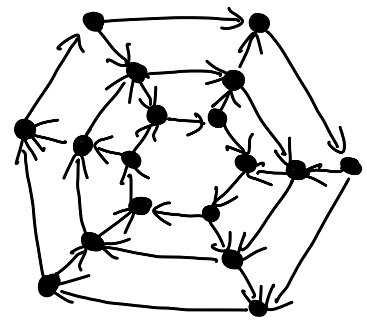
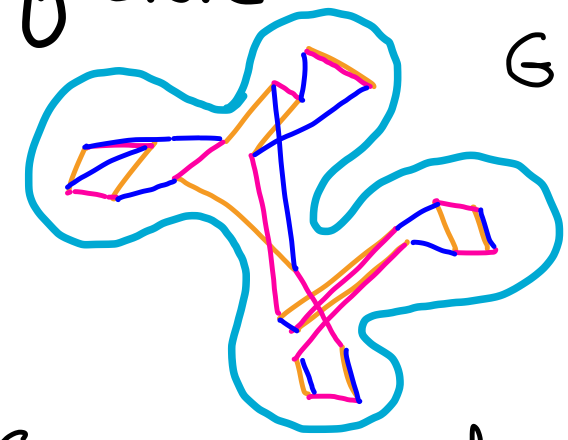


The Matching Grid Conjecture

G matching covered

⇝ connected and every edge belongs to a perfect matching

The Matching Grid Conjecture



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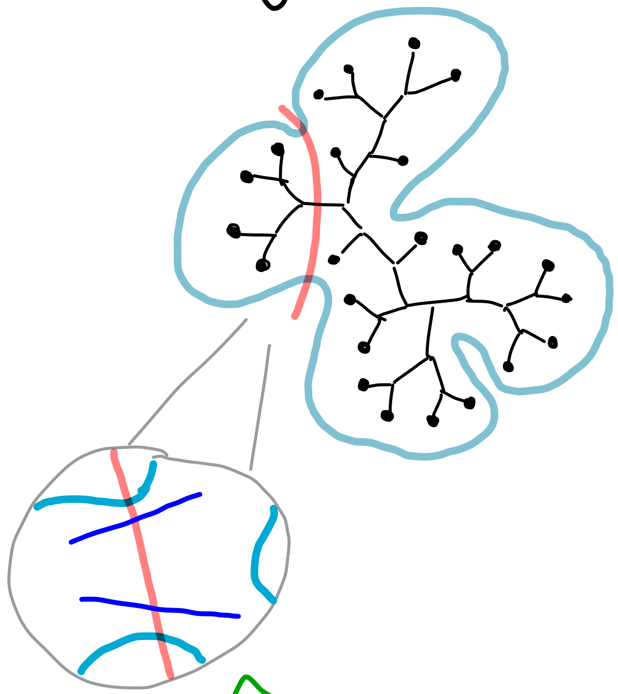
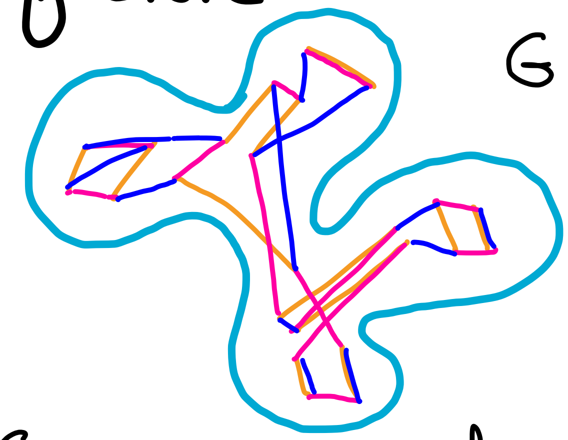
Perfect Matching Width



consider the maximum number of edges a perfect matching can have in the cut

The Matching Grid Conjecture



G matching covered

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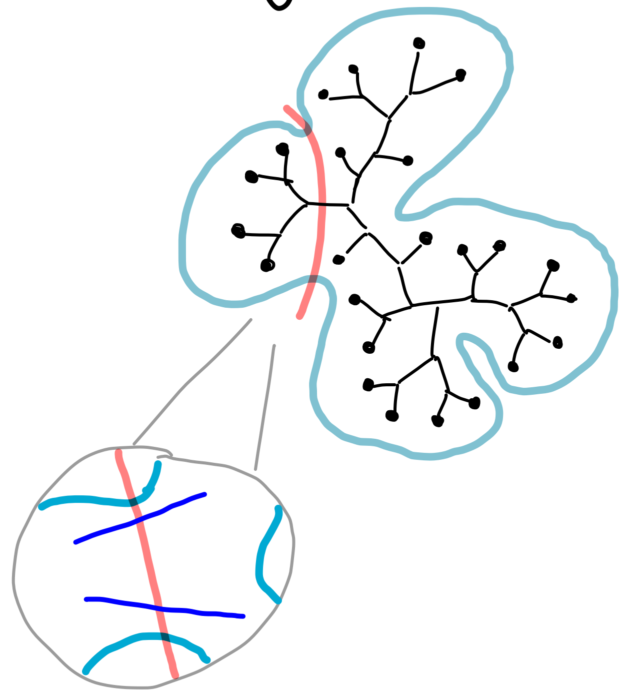
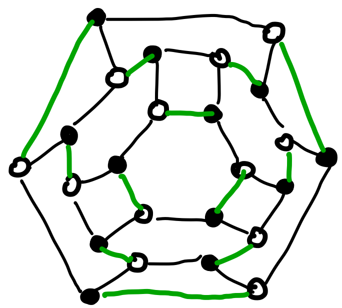


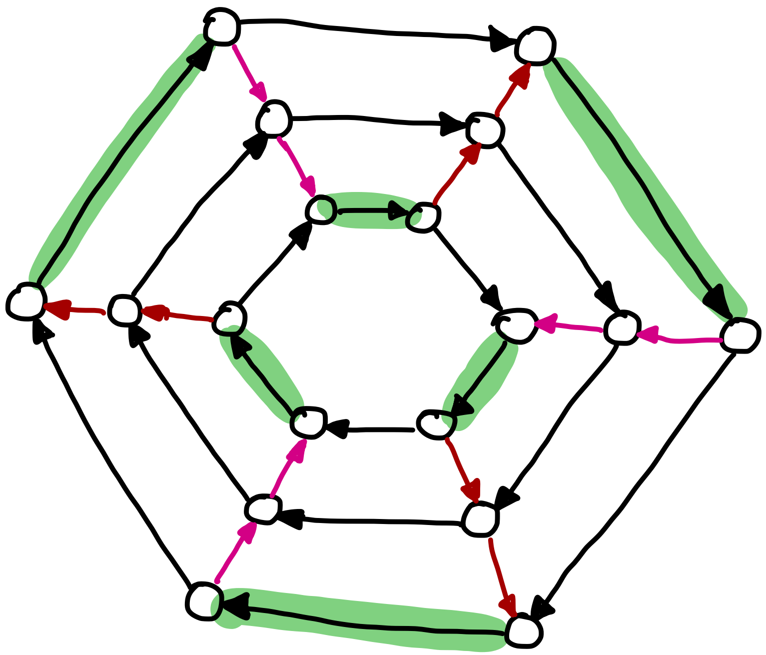
"well behaved" under matching minors

Conjecture (Norine)

A matching covered graph either has small perfect matching width or contains a matching grid as a matching minor.



Directed Graphs & Bipartite Matching Covered Graphs



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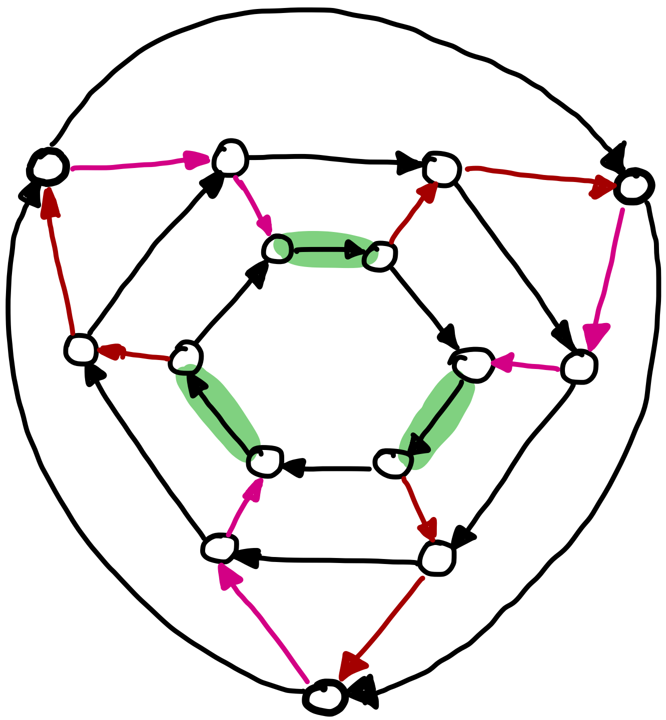

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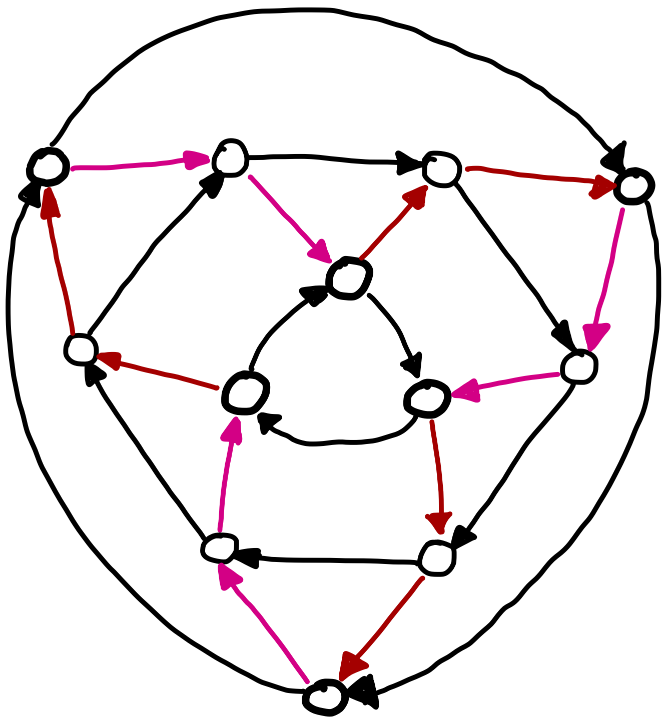
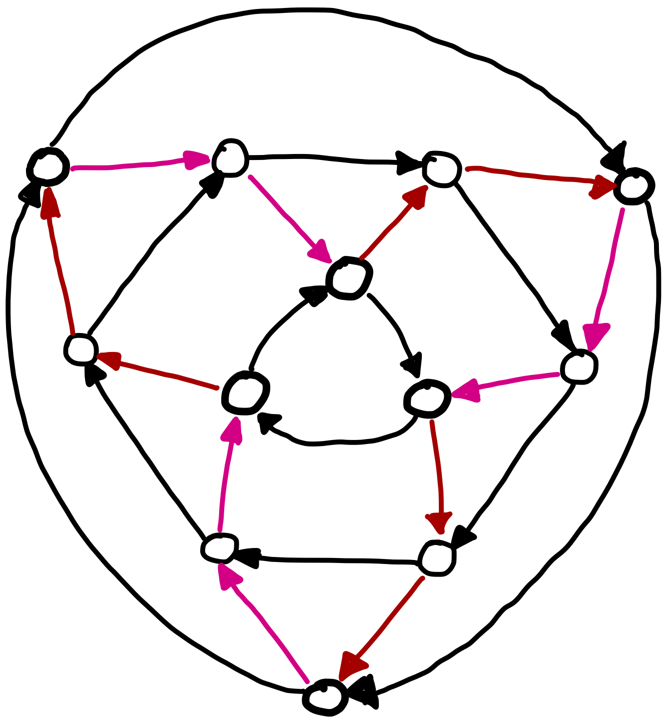


replace every vertex by an edge
and sort its incident edges by out-
and ingoing

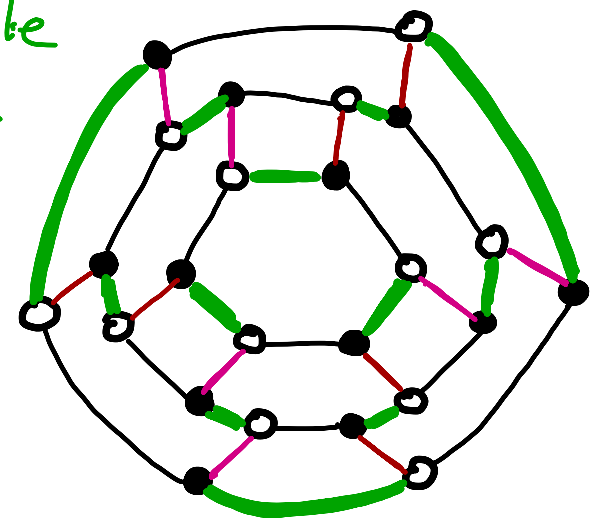
Directed Graphs & Bipartite Matching Covered Graphs

$D(G, M)$



yields a bipartite graph G with a marked perfect matching M

G



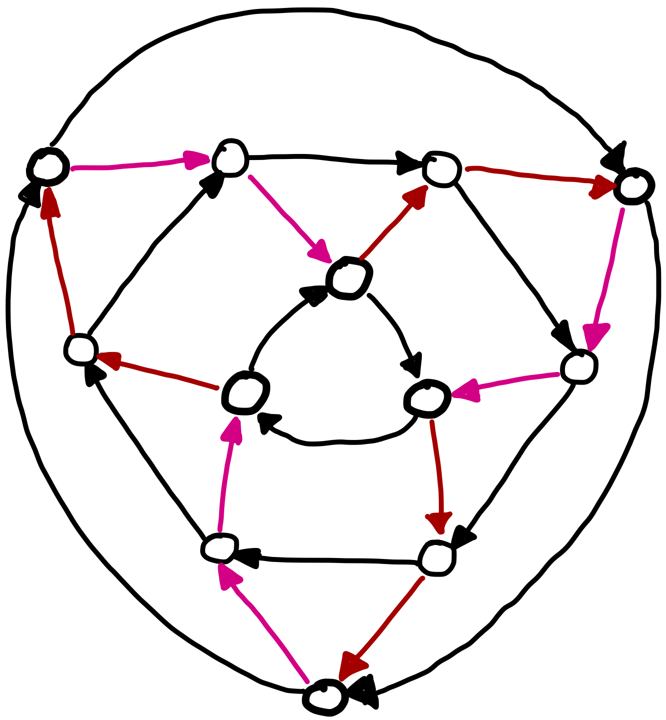
$\rightsquigarrow$ replace every vertex by an edge and sort its incident edges by out- and ingoing

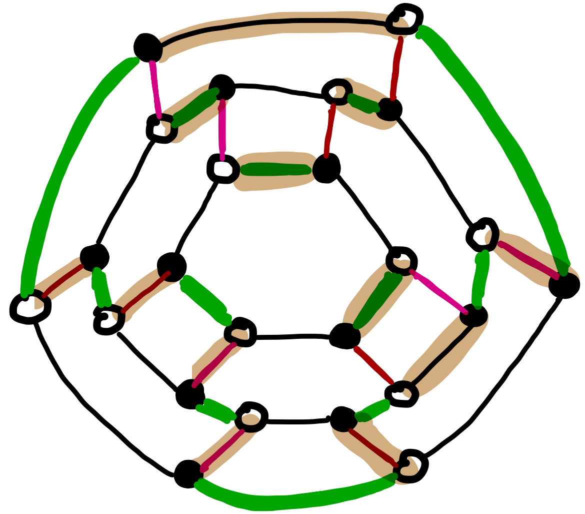
Directed Graphs & Bipartite Matching Covered Graphs

$D(G, M)$

works in both directions

G

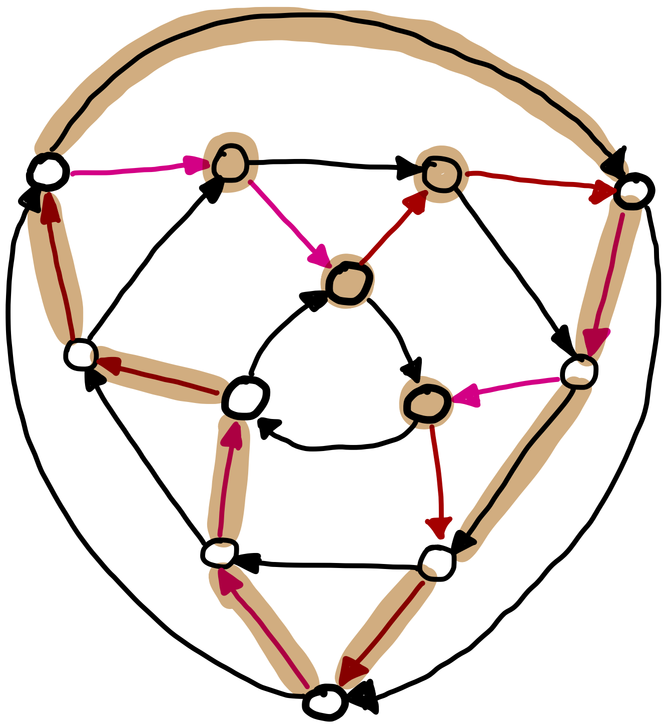


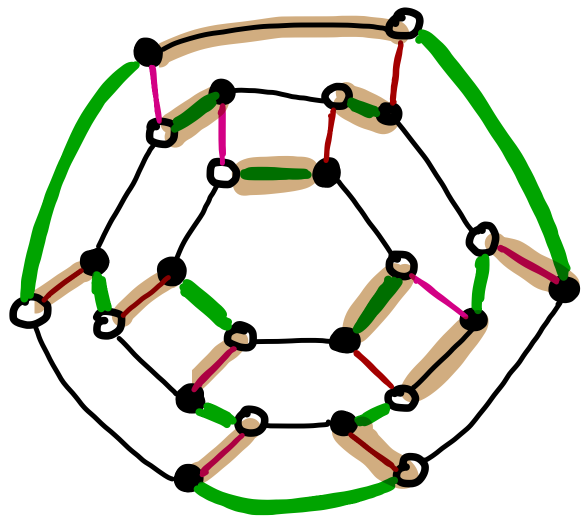


consider a p.m. M' different from M

Directed Graphs & Bipartite Matching Covered Graphs

$D(G, M)$ works in both directions G





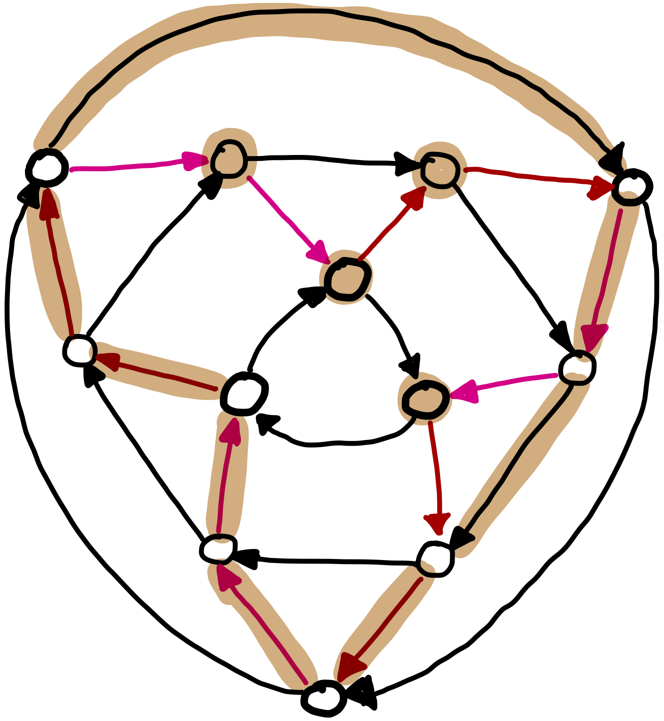
M' yields a family of pairwise vertex disjoint directed cycles ⟵ consider a p.m. M' different from M

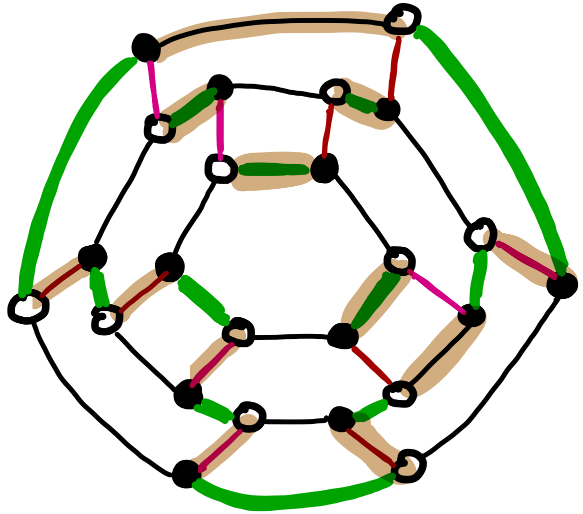
Directed Graphs & Bipartite Matching Covered Graphs

$D(G, M)$

works in both directions

G





Minors

Lemma (McCuaig)

$D(G,M)$ has $D(G',M')$ as a butterfly minor

$\iff$

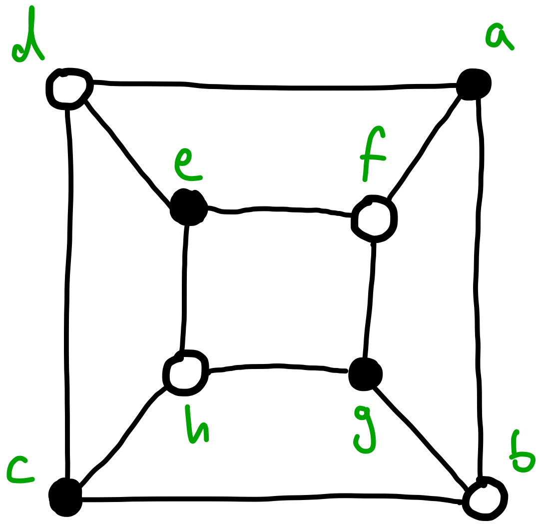
G has G' as a matching matching minor "that respects M"

Plan

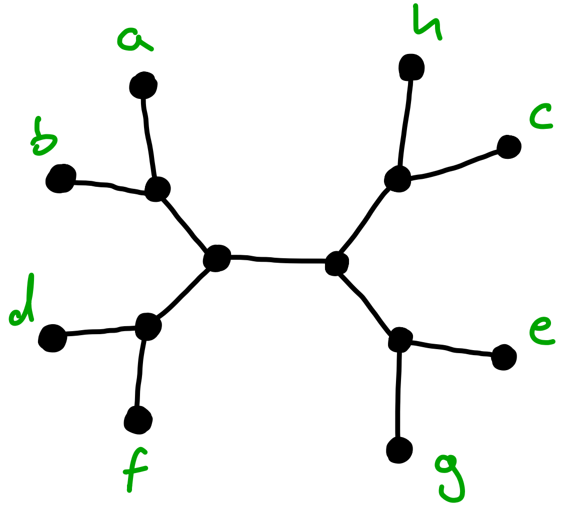
1) Link perfect matching width of bipartite matching covered graphs to directed treewidth.

2) Use the Directed Grid Theorem to settle Norines Conjecture for the bipartite case.

M-Perfect Matching Width

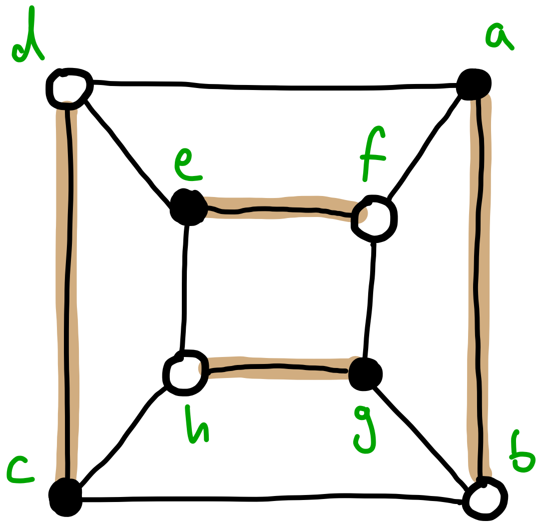


G



optimal perfect matching decomposition for G

M-Perfect Matching Width



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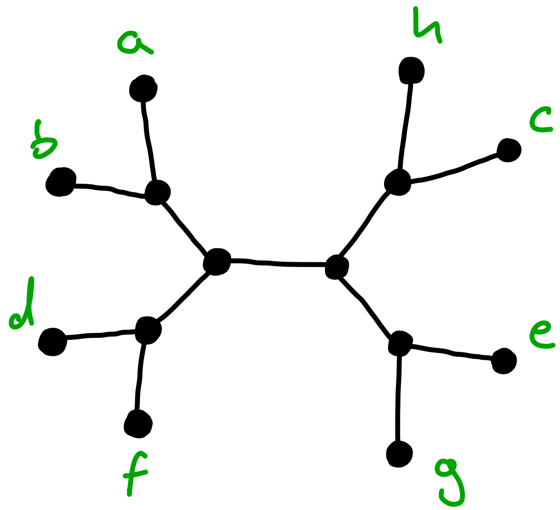


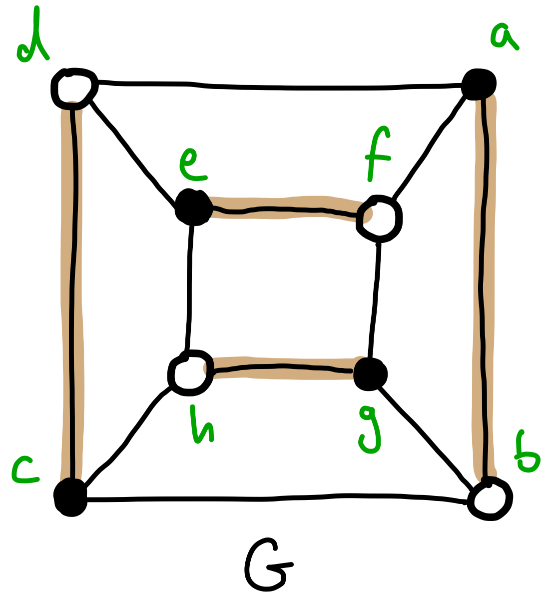
optimal perfect matching decomposition for G

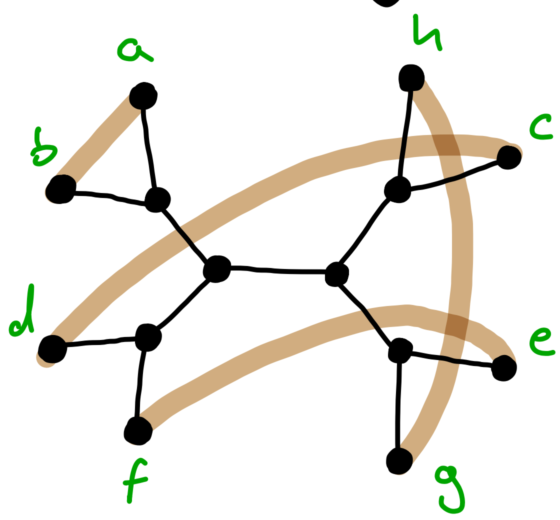
Every digraph corresponds to a pair:

G a bipartite graph

M a perfect matching of G

M-Perfect Matching Width





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Every digraph corresponds to a pair:

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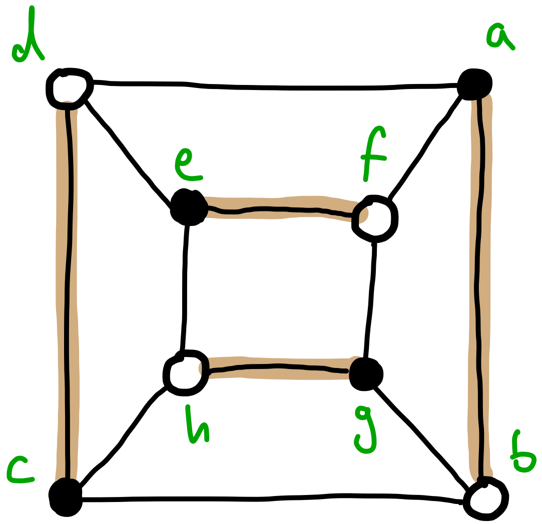
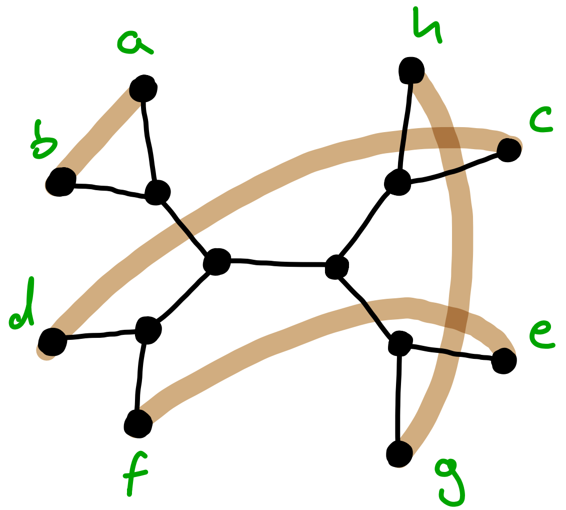
Problem: M might be "scattered" across an optimal decomposition

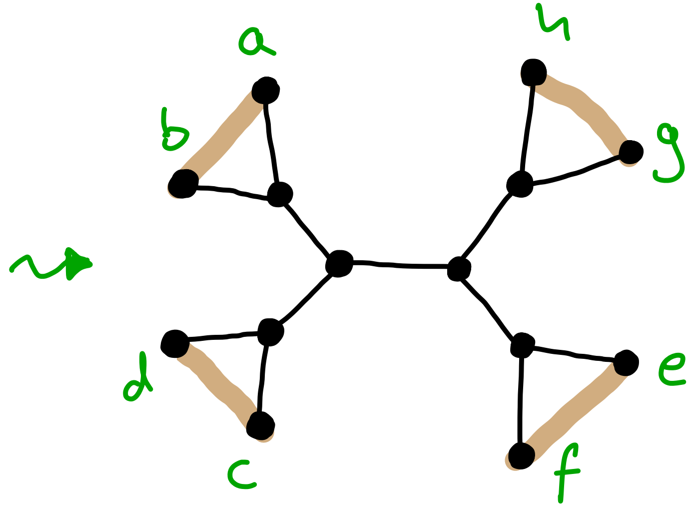
$\mathcal{M}$-Perfect Matching Width



G



optimal perfect matching decomposition for G



optimal $\mathcal{M}$-decomposition for G

Every digraph corresponds to a pair:

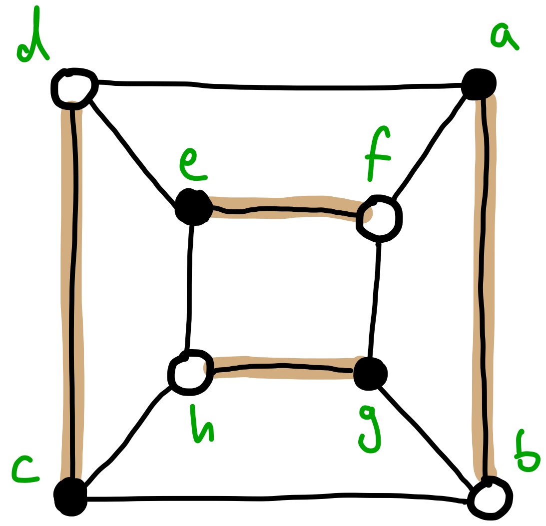
G a bipartite graph

$\mathcal{M}$ a perfect matching of G

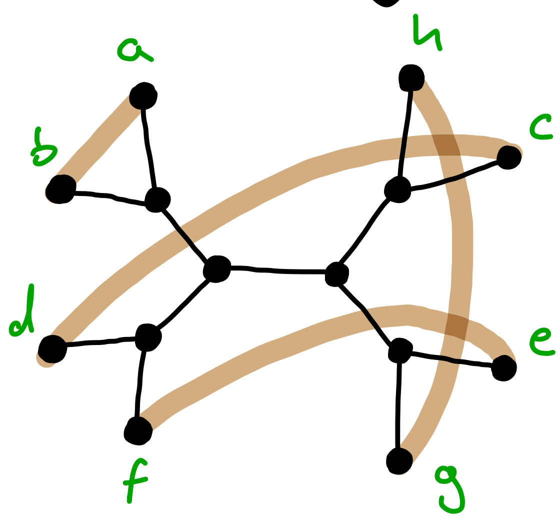
Problem: $\mathcal{M}$ might be "scattered" across an optimal decomposition

Solution: Only consider decompositions that respect $\mathcal{M}$ $\Rightarrow$ $\mathcal{M}$-pmw

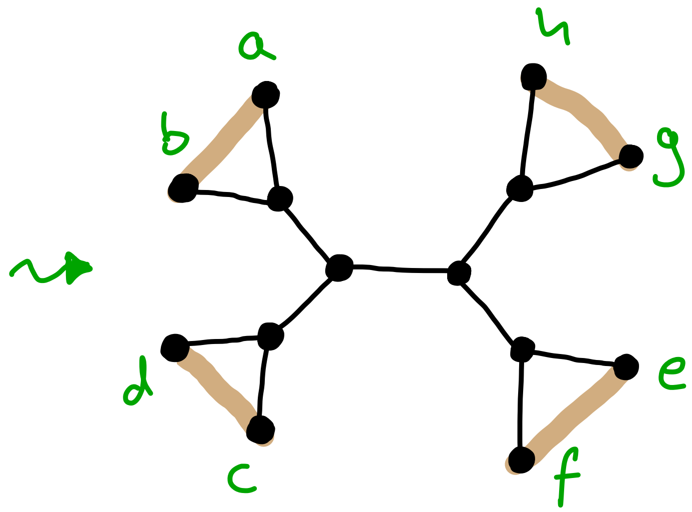
$\mathcal{M}$-Perfect Matching Width



G



optimal perfect matching decomposition for G

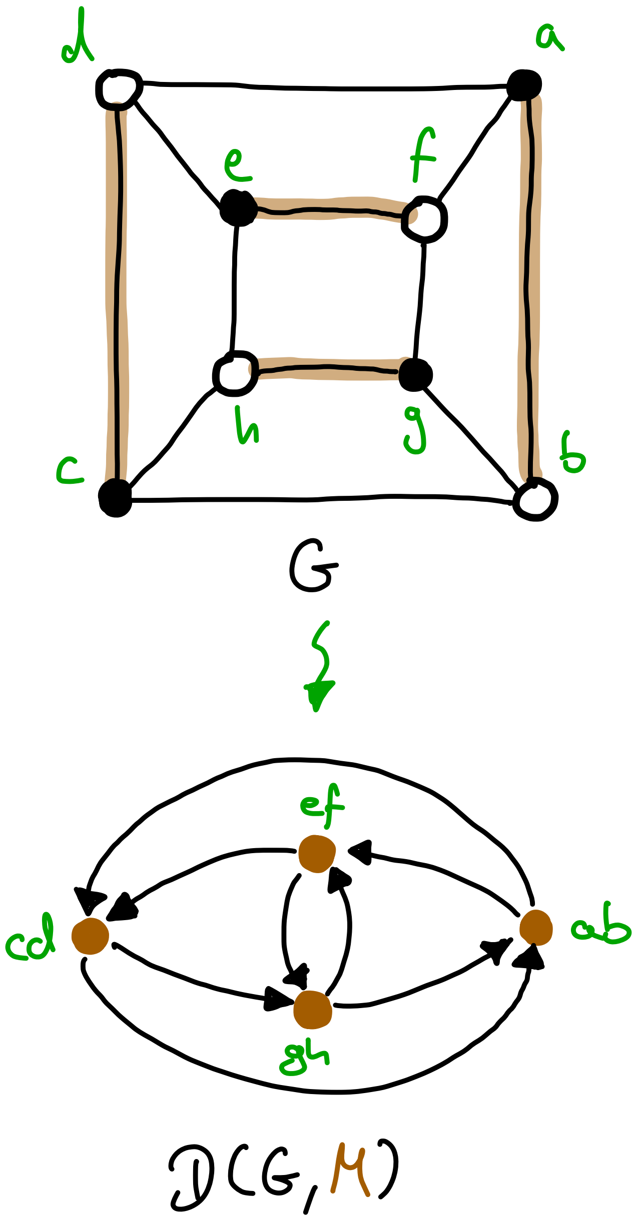


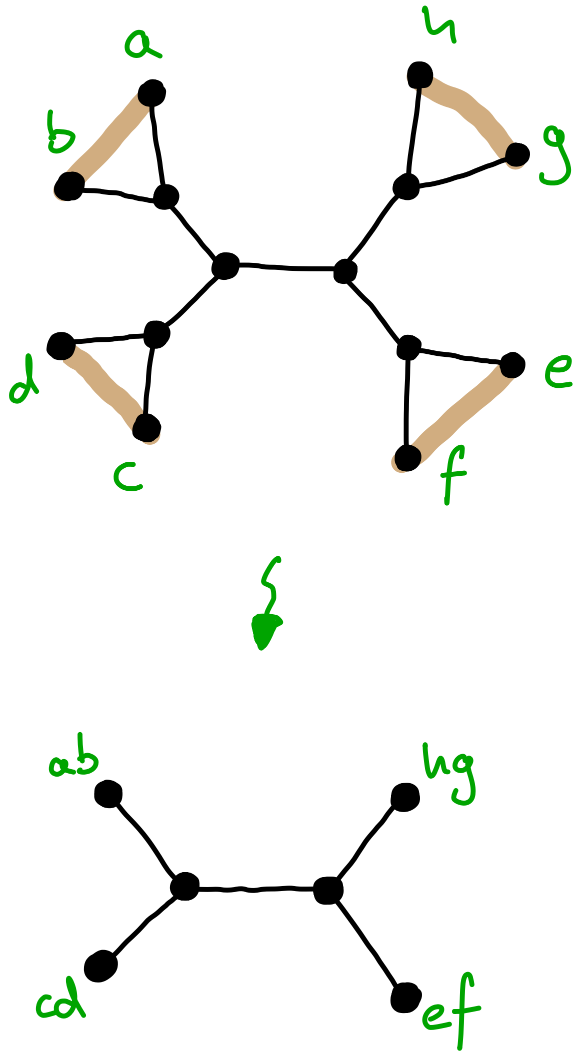
optimal $\mathcal{M}$-decomposition for G

Theorem

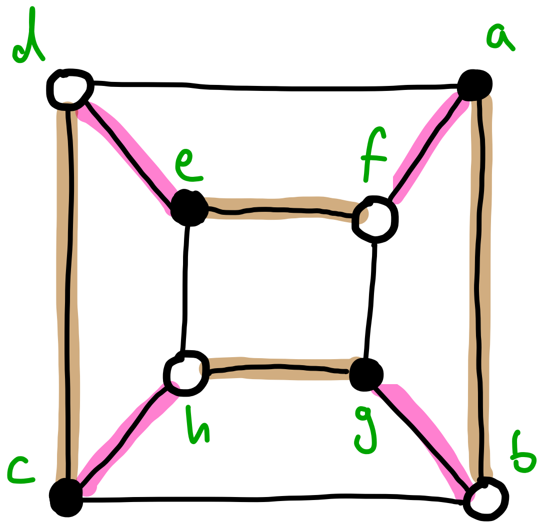
$$\mathrm{pmw}(G) \leq \mathcal{M}\text{-}\mathrm{pmw}(G) \leq 2\,\mathrm{pmw}(G)$$

Towards Digraphs





Towards Digraphs



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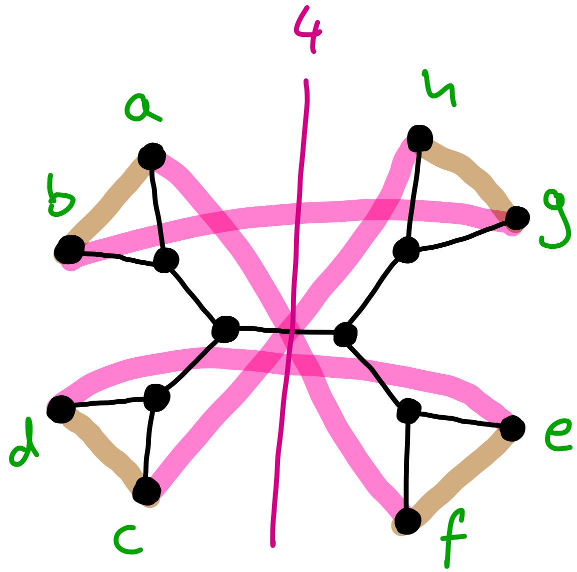
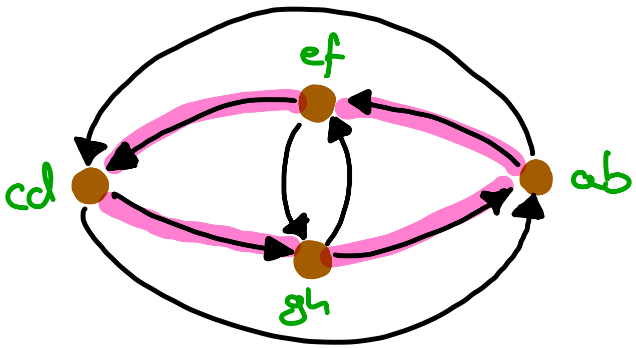
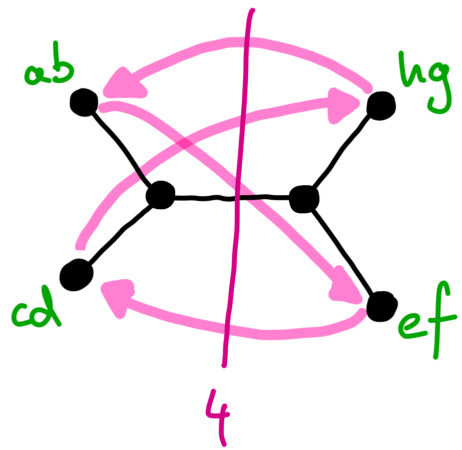


Width?

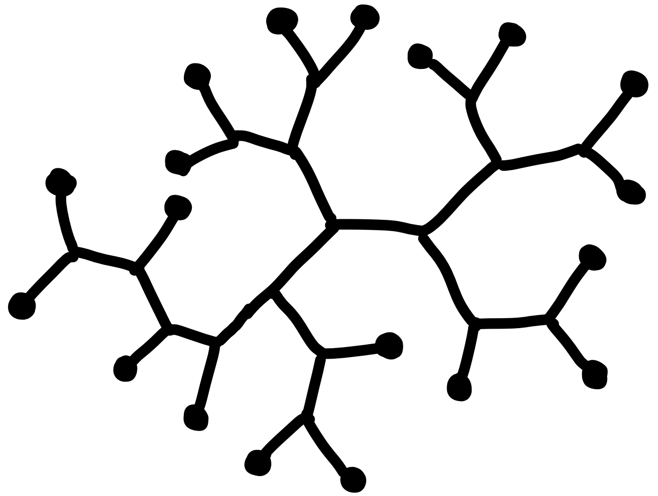


$\mathcal{D}(G, M)$



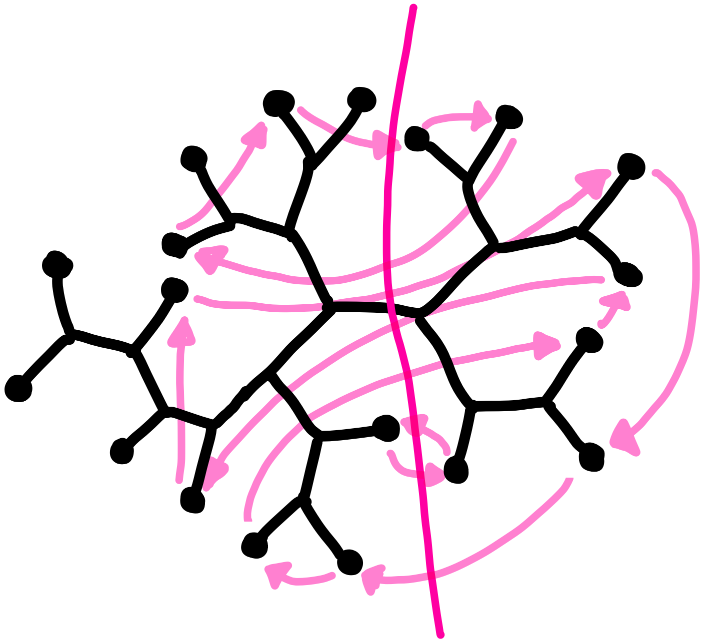
$\Rightarrow$ consider collections of vertex disjoint directed cycles

Cyclewidth



- cubic tree
- vertices of D mapped to leaves

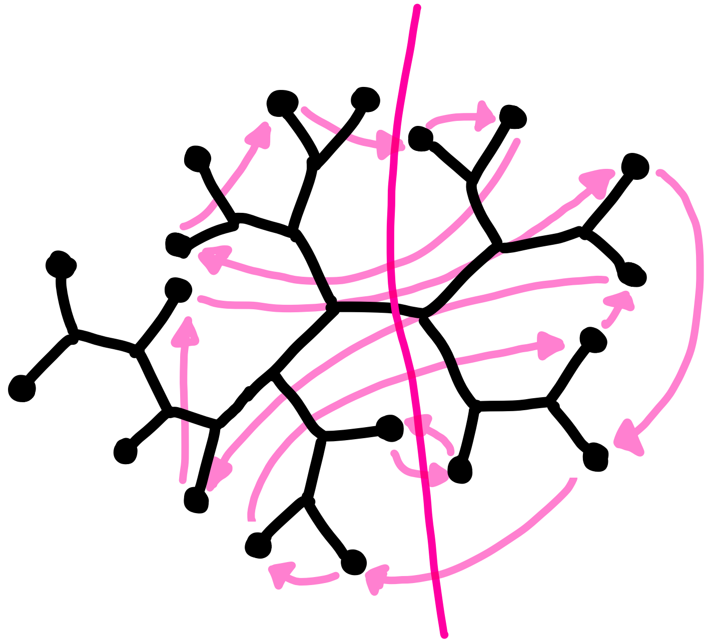
Cyclewidth



- cubic tree
- vertices of D mapped to leaves

- width is measured via the number of edges vertex disjoint cycle families have in the cuts

Cyclewidth



- cubic tree
- vertices of D mapped to leaves
- width is measured via the number of edges vertex disjoint cycle families have in the cuts

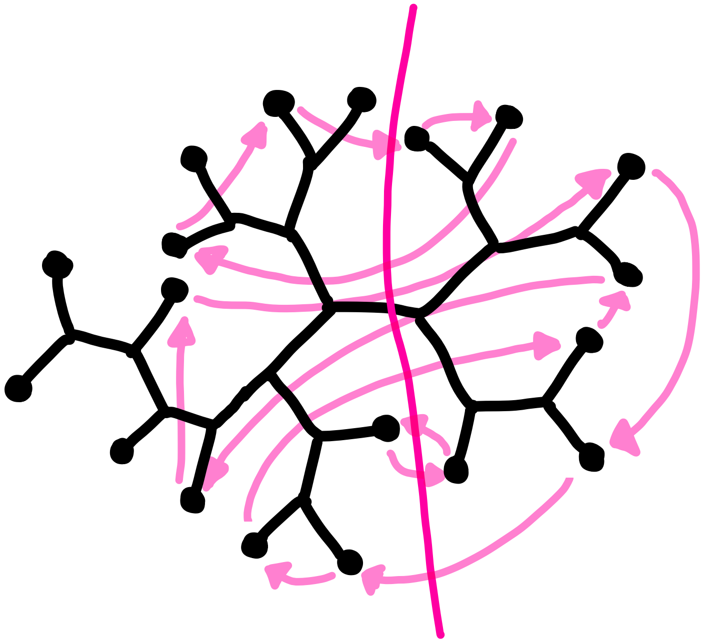
Lemma

Cylindrical grid has large cyclewidth

Lemma

$cyw(D) \leq 2\, dtw(D)$

Cyclewidth



- cubic tree
- vertices of D mapped to leaves
- width is measured via the number of edges vertex disjoint cycle families have in the cuts

Lemma

Cylindrical grid has large cyclewidth

Lemma

$\mathrm{cyw}(D) \leq 2\,\mathrm{dtw}(D)$

Theorem

A class of digraphs has bounded cyclewidth iff it has bounded directed treewidth.

Deducing a Grid Theorem

<u>Lemma</u>

Cyclewidth is closed under butterfly minors

$\Downarrow$ Grid Theorem for directed treewidth

Grid Theorem for cyclewidth

Deducing a Grid Theorem

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$$\mathrm{cyw}(\mathcal{D}(G,M)) = M\text{-}\mathrm{pmw}(G)$$

$$\Downarrow$$

$$\mathrm{pmw}(G) \leq \mathrm{cyw}(\mathcal{D}(G,M)) \leq 2\,\mathrm{pmw}(G)$$

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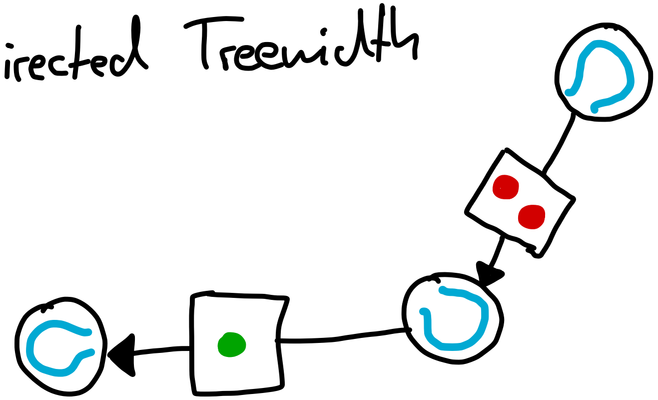
\+ McCuaigs Lemma on butterfly and matching minors

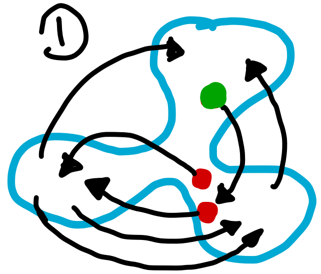
Grid Theorem for bipartite perfect matching width

Remarks on Cyclewidth

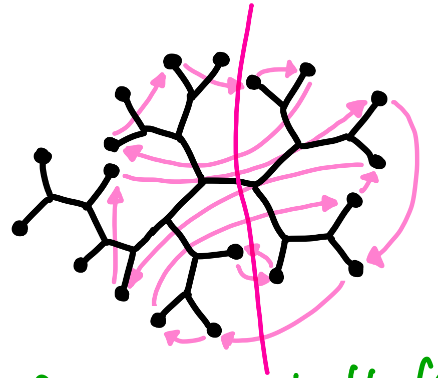
Directed Treewidth



"well behaved" under butterfly minors



Cyclewidth



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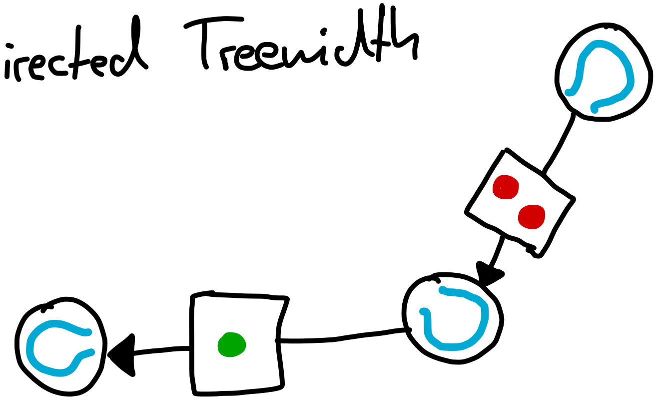

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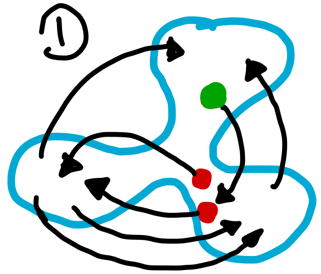

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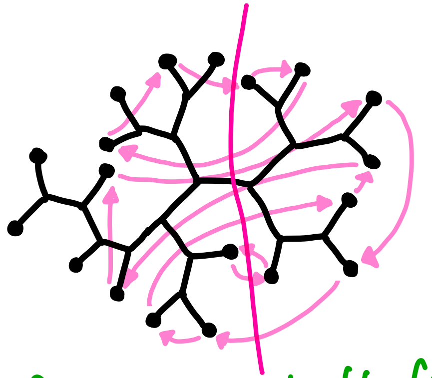

closed under butterfly minors

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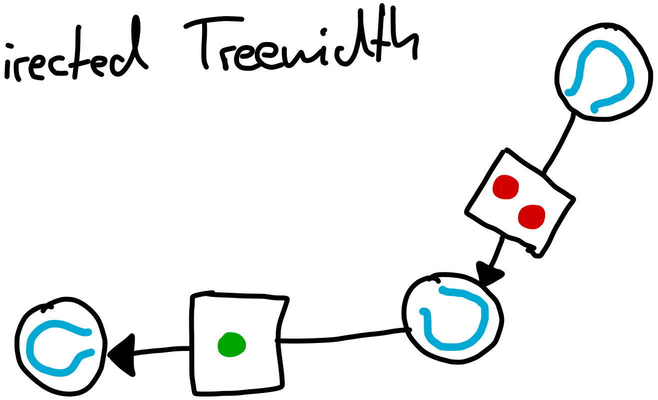
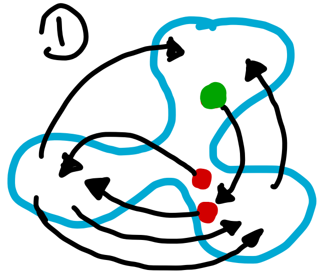
Remarks on Cyclewidth

Directed Treewidth





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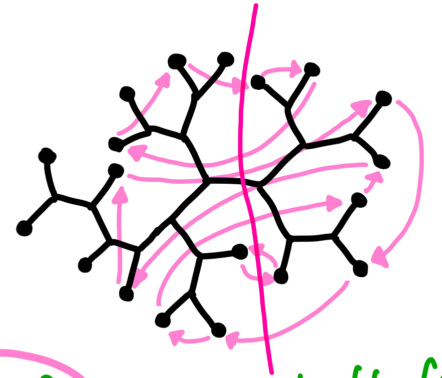

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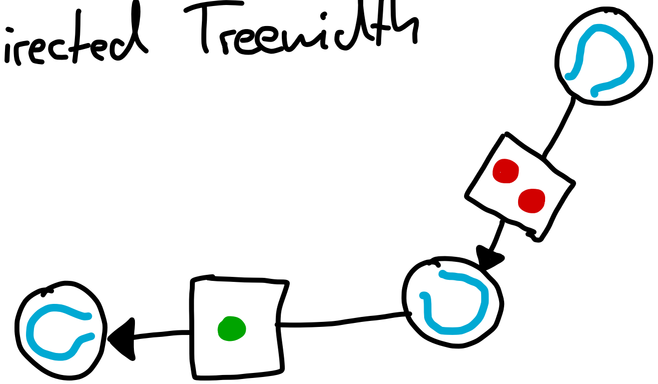
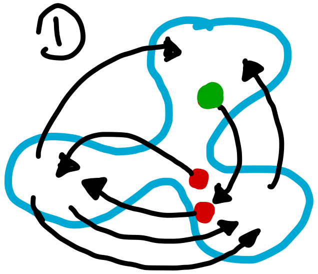
cyw is better behaved than dtw

Task: Characterise classes of small cyw by forbidden minors

Remarks on Cyclewidth

Directed Treewidth





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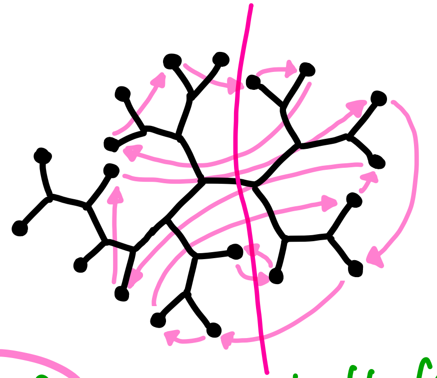

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Thank You